One crib sheet of US letter size, filled on its two sides, is allowed. Any calculator or computing device, including your cell phone, is not allowed. **Good luck!**

$$\begin{aligned}
\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\sin(2a) &= 2 \sin a \cos a \\
\cos(2a) &= 2\cos^2 a - 1 \;=\; 1 - 2\sin^2 a \\
\sin\delta &= \delta + O(\delta^3) \\
\cos\delta &= 1 - \frac{1}{2}\delta^2 + O(\delta^4) \\
(1+\delta)^\alpha &= 1 + \alpha\delta + \frac{1}{2}\alpha(\alpha-1)\delta^2 + O(\delta^3) \\
\log(1+\delta) &= \delta - \frac{1}{2}\delta^2 + O(\delta^3) \\
e^\delta &= 1 + \delta + \frac{1}{2}\delta^2 + O(\delta^3)
\end{aligned}$$

In the above, δ is assumed to be small: $|\delta| \ll 1$.

Show all your work. Your name and page numbers must be clearly written on your solution sheets. Be neat in writing. Partial credit for an incorrect answer will be given whenever there is a good reason for it (e.g. correct logical steps from an incorrect previous answer). Very little credit may be given for a correct answer, if not properly derived/explained. Each problem is worth 100 points.

Problem 1 A compass needle with magnetic moment $\vec{M}$ is free to rotate in a constant magnetic field $\vec{B}$. Both these vectors can be considered to be in the same plane. The potential energy is given by $U = -\vec{M}\cdot\vec{B}$ and the moment of inertia of the compass needle is given by I. (a) Make a sketch of the potential energy as a function of the angle θ between $\vec{M}$ and $\vec{B}$. In your sketch, you must indicate at least two extrema (minimum or maximum) of the potential energy for $|\theta| < 2\pi$. (b) Prove that the small angle motion ($|\theta| \ll 1$) is a simple harmonic motion. (c) Find the natural frequency of the simple harmonic motion as a function of I, $B \equiv |\vec{B}|$, and $M \equiv |\vec{M}|$.

Problem 2 A ball of mass m is thrown upwards with initial speed v_0. The magnitude of the downward gravitational force is given by mg, where g is a positive constant. The magnitude of the air resistance is given by

$$|\vec{F}_{a.r.}| = mk(v + \beta v^2),$$

and the direction of the air resistance is opposite to the velocity. k and β are positive constants. We shall assume that the air resistance can be treated as a perturbation: k (or more precisely, dimensional parameter $\propto k$) can be regarded as small. To the leading order correction by the perturbation, find

(a) the speed at which the ball comes back to the original height, and

(b) the maximum height that the ball reaches.

For both parts, the answer should be expressed as the zeroth order answer times (1 + small term), where the small term, which depends on both β and k as well as other parameters, is a dimensionless expression proportional to k^n, assuming the leading order correction comes from the n-th order perturbation. [Hint: For both parts, use the argument based on energy consideration.]

Problem 3 A simple harmonic oscillator with no damping is subjected to an external force that involves *two* frequencies, ω and 2ω, in the following manner.

$$m\ddot{x} = -kx + mB\cos(\omega t) + mB\sin(2\omega t),$$

where B is a positive constant and

$$2\omega = \sqrt{\frac{k}{m}}.$$

Find the *general solution* $x(t)$ for this problem. [Hint: The superposition principle.]

Problem 4 Two massless springs are connected in series. The spring constant of the first spring is given by k_1, and the spring constant of the second spring is given by k_2. The change of the length of the first spring relative to its equilibrium length is given by x_1, and that of the second spring is given by x_2. The first spring is fixed to a wall on one end, while its other end is connected to the second spring in series. At the other end of the second spring is attached a mass m, which slides on the surface of a table without friction. The mass moves only along a specific direction (say the x direction), which coincides with the direction along which the two springs are connected in series. We shall start by treating x_1 and x_2 as *two independent* generalized coordinates of this problem.

(a) Find the total potential energy, U, of the system, as a function of k_1, k_2, x_1, and x_2.

(b) Find the kinetic energy, K, of the system, as a function of m, $\dot{x}_1$, and $\dot{x}_2$.

(c) Find the Lagrangian, $L = K - U$.

(d) The Lagrangian $L = L(x_1, x_2, \dot{x}_1, \dot{x}_2)$ is a function of four independent variables, x_1, x_2, $\dot{x}_1$, and $\dot{x}_2$. Apply $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}$ to the $i = 1$ case and the $i = 2$ case, to find *two* coupled equations of motion for x_1 and x_2.

(e) From the equations of motion, show that $k_1 x_1 = k_2 x_2$ at all times, as you would have expected from Newton's third law.