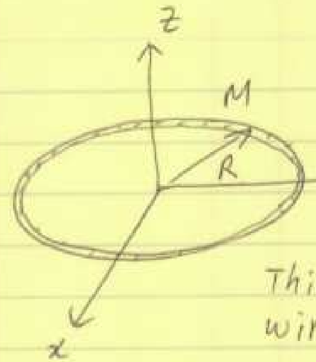
Notes for Lecture 19

Inertia Tensor, Non-linear Physics

19.1 Inertia tensor

Before going on to the non-linear physics, let us look at some typical inertia tensors and their derivations.

①



Thin circular wire

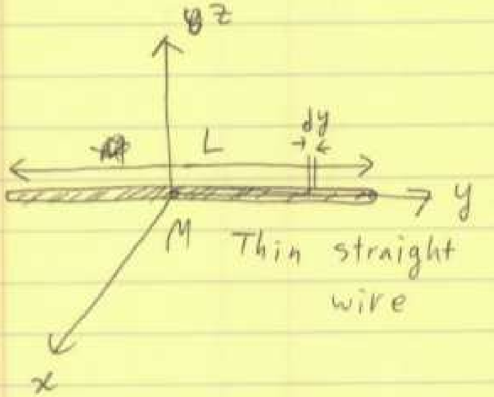
$$\overleftrightarrow{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$\begin{cases} I_1 + I_2 = I_3 \\ I_1 = I_2 \end{cases}$$

$$I_3 = MR^2$$

$$\therefore \overleftrightarrow{I} = \frac{1}{2} \begin{bmatrix} MR^2 & 0 & 0 \\ 0 & MR^2 & 0 \\ 0 & 0 & 2MR^2 \end{bmatrix}$$

②



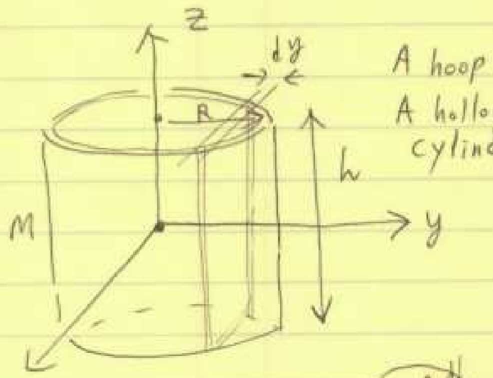
Thin straight wire

$$\begin{cases} I_1 + I_2 = I_3 \\ I_2 = 0 \end{cases}$$

$$I_3 = \int_{-\frac{L}{2}}^{\frac{L}{2}} dm\, y^2 \qquad dm = \frac{dy}{L} \cdot M$$

$$= \frac{1}{12} ML^2$$

③



A hoop
A hollow cylinder

$$I_3 = MR^2$$

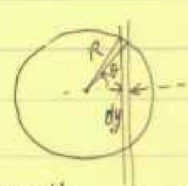
$$I_1 = I_2$$

$$dm = \frac{d\theta}{2\pi} \cdot 2 \cdot M = \frac{M}{\pi} \frac{dy}{R\sqrt{1 - \frac{y^2}{R^2}}}$$

$$dI_1 = \frac{1}{12} dm\, h^2 + dm\, y^2$$

(parallel axis theorem)

Top view



$$R\cos\theta = y$$

$$-R\sin\theta\, d\theta = dy$$

$$d\theta = \frac{-dy}{R\sin\theta}$$

$$I_1 = \frac{M}{\pi}\int_{-R}^{R} dy \cdot \frac{\frac{1}{12}h^2 + y^2}{\sqrt{R^2 - y^2}} = \frac{2M}{\pi}\int_0^R dy \cdot \frac{\frac{1}{12}h^2 + y^2}{\sqrt{R^2-y^2}}$$

$y = R\cos\theta \rightarrow \sqrt{R^2-y^2} = R\sin\theta$

$\theta = \frac{\pi}{2} \sim 0 \qquad y^2 = R^2\cos^2\theta = R^2 \cdot \frac{1+\cos(2\theta)}{2}$

$dy = -R\sin\theta \, d\theta$

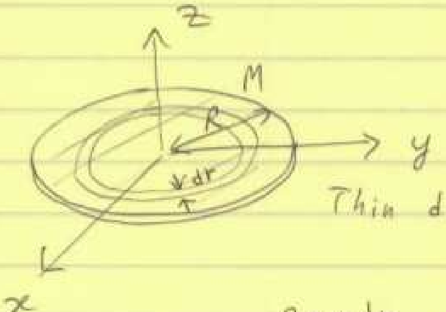
$$\Rightarrow I_1 = \frac{2M}{\pi}\int_0^{\frac{\pi}{2}} d\theta \left\{\frac{1}{12}h^2 + \frac{R^2}{2}\left(1+\cos(2\theta)\right)\right\}$$

$$= M\left(\frac{1}{12}h^2 + \frac{1}{2}R^2\right) = I_2$$

Note that this goes to $\frac{1}{12}Mh^2$ if $R \to 0$
(correct as in ②)

to $\frac{1}{2}MR^2$ if $h \to 0$
(correct as in ①)

④



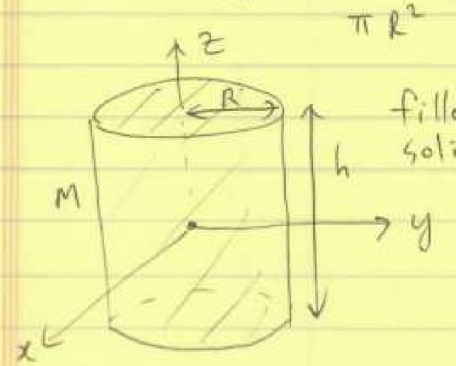
$I_1 = I_2 = \frac{1}{2}I_3$

$I_3 = \int_0^R dm\, r^2$

$dm = \frac{2\pi r\, dr}{\pi R^2} \cdot M$

$I_3 = \frac{1}{2}MR^2$

⑤



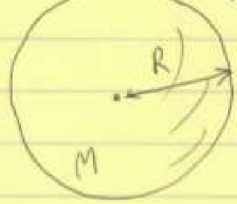
filled cylinder
solid

$I_3 = \frac{1}{2}MR^2$

$I_1 = I_2 = M\left(\frac{1}{12}h^2 + \frac{1}{4}R^2\right)$

calculate ?!
Or, learn from ①,②,③,④.

(6) Sphere (filled) (solid)



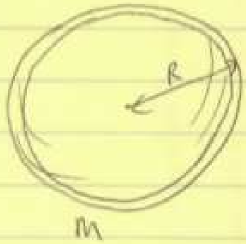
$$I_1 = I_2 = I_3$$

$$I_1 + I_2 + I_3 = 2\int_0^R dm\, r^2 \qquad dm = \frac{4\pi r^2\, dr}{\frac{4\pi}{3} R^3}\cdot M$$

$$= \frac{6}{5} M R^2$$

$$\therefore\ I_1 = I_2 = I_3 = \frac{2}{5} M R^2$$

(7) Sphere (shell) ("empty")

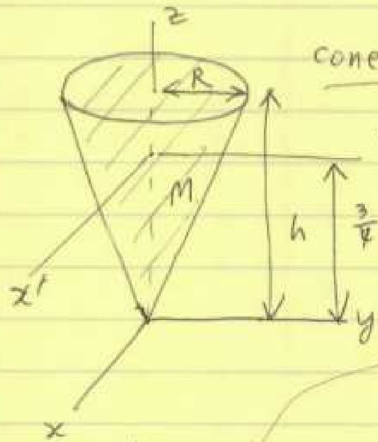


$$I_1 = I_2 = I_3$$

$$I_1 + I_2 + I_3 = 2\int_M dm\, R^2$$

$$\therefore\ I_1 = I_2 = I_3 = \frac{2}{3} M R^2$$

(8) cone (solid)



First in the xyz coords.

$$I_1 = I_2.$$

From (4)

$$I_3 = \int_0^h \frac{dz\ R^2\left(\frac{z}{h}\right)^2\cdot\pi}{\pi R^2\cdot\frac{h}{3}}\cdot M\cdot\frac{R^2\left(\frac{z}{h}\right)^2}{2}$$

$$= \frac{3}{10} M R^2$$

$$I_1 = M\int_0^h dz\cdot\frac{3z^2}{h^3}\cdot\left\{z^2 + \frac{1}{4}R^2\left(\frac{z}{h}\right)^2\right\} = M\left(\frac{3}{5}h^2 + \frac{3}{20}R^2\right)$$

(4) and the P. axis theorem.

$$I_1' = I_1 - M\left(\frac{3}{4}h\right)^2 = M\left(\frac{3}{80}h^2 + \frac{3}{20}R^2\right)$$

(P. axis theorem)

19.2 Non-linear systems

19.2.1 Phase space, 1st order ODE

Newton's equation is a 2nd order ODE in the real space. However, in the phase space it is a 1st order ODE. Why? Because the equation

$$m\ddot{\vec{r}} = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$$

becomes a set of equations

$$\begin{aligned} \dot{\vec{v}} &= \vec{F}(\vec{r}, \vec{v}, t)/m \\ \dot{\vec{r}} &= \vec{v} \end{aligned}$$

In this new representation, Newton's law becomes a first order equation in the $(\vec{r}, \vec{v})$ space. Generalization of this new representation to a multi-particle case is straightforward: just take $\vec{r}$ to include all position coordinates of all particles, and $\vec{v}$ to include all velocity components of all particles.

With the language of the phase space, and the above set of equations in the phase space, what we discussed about Newtonian determinism becomes very lucid. It means that given an initial condition, a point is specified in the phase space. Then, the time evolution of that point is dictated by Newton's 2nd law as a mapping from the initial point to the next point at the next time (dt) and then at a later time ($2dt$) and so on and so forth. In this sense, **classical dynamics can be thought of as a mapping of a point in phase space**.

The numerical integration of Newton's EOM usually sets up the equation in phase space, as shown above, and do a single integration, using, e.g., the Runge-Kutta method.

19.2.2 Linear and non-linear systems

In a linear system, if two slightly different inputs go in, then "slightly different outputs" come out. Here, slightly different outputs mean the outputs whose difference is linearly proportional to the initial difference.

This sounds reasonable. But, let us be a bit more analytical.

Consider dividing up time into many tiny steps of Δt, which is much smaller than the experimental resolution. Newton's EOM can then be written as a succession of discrete transformations:

$$\vec{P}_{n+1} = \vec{P}_n + (\vec{v}_n, \vec{F}(\vec{P}_n, t_n))\Delta t$$

$\vec{P}$ is a point in phase space, $(\vec{r}, \vec{v})$, and $n = 0, 1, 2, \cdots$. $\Delta t = t/N$, where N is a large natural number, and $t_n = n\Delta t$. Taking this recurrence relation, a computer can use this equation to find the solution at any specified time t in the future or in the past. Note that Δt here is basically dt, but I am using Δt for notational convenience.

The function $\vec{F}(\vec{P}, t)$ can be written as

$$\vec{F} = \vec{f}(t) + \vec{g}(\vec{P}, t)$$

where $\vec{f}(t)$ is a term that is purely dependent on time (like the driving force term of a driven SHO), and $\vec{g}$ is a function that explicitly depends on dynamical variables $(\vec{P})$ with a possible dependence on t as well.[1]

> A linear system is defined as a system where $\vec{g}(\vec{P}, t)$ is a linear function of $\vec{P}$.

The SHO (damped or driven) we discussed so far is a linear system. In 1D, $\vec{g}(\vec{P}) = -kx - bv$. This is a linear function on a phase space vector $\binom{x}{v}$.

Now, let us look at the mathematics of the transformation from $\vec{P}_n$ to $\vec{P}_{n+1}$ for a linear system. We start from the initial condition $\vec{P}_0$.

$$\vec{P}_1 = \vec{P}_0 + (\vec{v}_0, \vec{F}(\vec{P}_0, 0))\Delta t$$

Say, we have a slightly different initial condition $\vec{P}_0{}' = \vec{P}_0 + \varepsilon\vec{d}$, with $\vec{d} = (\vec{r}_d, \vec{v}_d)$, and ε is a very small number. What is the result of the transformation for $\vec{P}_0{}'$?

[1]An example of this is a person who pumps a swing by simply sitting up and down to change a parameter (l) of the system. This mechanism of pumping a swing is called the "parametric resonance" and it is different from the resonance that we discussed in the previous section. Please look it up if you are interested.

$$\begin{aligned}
\vec{P}_1{}' &= \vec{P}_0 + \varepsilon\vec{d} + (\vec{v}_0 + \varepsilon\vec{v}_d, \vec{F}(\vec{P}_0 + \varepsilon\vec{d}, 0))\Delta t \\
&= \vec{P}_0 + \varepsilon\vec{d} + (\vec{v}_0 + \varepsilon\vec{v}_d, \vec{F}(\vec{P}_0) + \varepsilon\vec{g}(\vec{d}, 0))\Delta t \qquad \vec{F}(\vec{P}_0{}', 0)) = \vec{F}(\vec{P}_0, 0) + \varepsilon\vec{g}(\vec{d}, 0) \\
&= \vec{P}_1 + \varepsilon\left[\vec{d} + (\vec{v}_d, \vec{g}(\vec{d}, 0))\Delta t\right]
\end{aligned}$$

Now, we will define $\vec{d}_0 \stackrel{def}{=} \vec{d}$, and define a recurrence relation for the *"difference vector* (up to a multiplicative constant ε)" $\vec{d}_n$, as follows.

$$\vec{d}_{n+1} = \vec{d}_n + (\vec{v}_n, \vec{g}(\vec{d}_n, t_n))\Delta t$$

Note that the difference vector evolves in time only by $\vec{g}$. This is reasonable since the effect of $\vec{f}$ is does not depend on $\vec{P}$.

So, what we have is the following.

$$\begin{aligned}
\vec{P}_0{}' &= \vec{P}_0 + \varepsilon\vec{d}_0 \\
\vec{P}_1{}' &= \vec{P}_1 + \varepsilon\vec{d}_1 \\
\vec{P}_2{}' &= \vec{P}_2 + \varepsilon\vec{d}_2 \qquad \text{Repeat what we did for } n = 0 \to 1. \\
&\cdots \\
\vec{P}_n{}' &= \vec{P}_n + \varepsilon\vec{d}_n \qquad \text{for any } n.
\end{aligned}$$

What is important to realize is that in this series, each step is *exact*. If we had a non-linear system, then a step might be approximately of the same form as what we wrote down here, but a cumulative effect of all steps may be completely non-linear in ε. What we showed here amounts to saying that such cannot be the case for a linear system, due to the exact linear nature of each step that is investigated above.

So, we have proved the following important fact.

> For a linear system, a small difference in the initial condition produces a difference in $\vec{P}$, which is only linearly proportional to the initial difference.

This is what we mean when we casually say "a small difference in inputs means a small difference in outputs." For instance, two balls rolled in slightly different directions will separate from each other with a very large difference in the positions as time becomes very large. But, the point is that such a difference will still be linear in the initial difference, and so it is "easy for us to understand it." If we consider a bound motion, such as two identical SHOs with slightly different initial conditions, such as slightly different phases (positions), then it is obvious that at all times the difference in positions will remain very small ($x_1(t) = A\cos(\omega t)$ and $x_2(t) = A\cos(\omega t + \varepsilon)$: $x_1 - x_2 \approx \varepsilon A \sin(\omega t) = O(\varepsilon)$ at *all times*). Why is this? Well, we can look at the above equation that we derived: $\vec{P}_n{}' = \vec{P}_n + \varepsilon \vec{d}_n$. If ε is vanishingly small, and if we have a bound motion (all $\vec{P}, \vec{P}', \vec{d}$ vectors are finite in size), then we can immediately see the following property.

> Consider two linear systems undergoing bound motions. Assume that the two systems are identical to each other, except for a vanishingly small difference in the initial conditions. Then, at any time, the difference between the two motions is vanishingly small, i.e. they are represented by two nearly identical points in the phase space at any given time.

The very defining characteristics of **chaos** is that this property breaks down. A seemingly negligible difference in the initial conditions between two bound state motions causes completely different results at a later time! Two almost identical inputs go in, and completely different results can come out, since the results exponentially diverge from each other, not linearly. This **sensitive dependence on initial conditions** is generally called chaos, while a more quantitative definition will be given later.

Some non-linear systems show chaotic behaviors. Non-linear problems are generally hard to solve by hand, and chaotic problems are, by definition, impossible to solve by hand. So the computer plays an important role in this topic.

19.2.3 Non-linear oscillations

For a real potential function that is encountered,[2] the deviation from the quadratic behavior of a SHO is more a rule than an exception. Suppose that there is a potential minimum at $x = 0$. In general, then, one should write,

$$U(x) = U_0 + \frac{1}{2} m\omega_0^2 x^2 + \frac{1}{3} m\alpha x^3 + \frac{1}{4} m\beta x^4 + \cdots$$

$$F(x) = -m\omega_0^2 x - m\alpha x^2 - m\beta x^3 + \cdots$$

[2] E.g. the Lennard- Jones potential.

When the α term and higher are not negligible, we have "non-linear oscillations." $F(x)$ is clearly non-linear with those terms included: for example, $(ax_1 + bx_2)^2 \neq ax_1^2 + bx_2^2$!

Every oscillation in nature is non-linear, as a rule. For instance, the reason why things tend to expand when they get hot is because of the non-linear oscillations inside materials (we can see the qualitative reason using classical mechanics, as we will do shortly).

It is reasonable to do a perturbation expansion on non-linear terms to get their leading order correction. This is done in Landau § 28, with the amplitude of the small oscillation as the perturbation parameter. Here, we will not pause to derive[3] the following result. However, it is of interest to understand the qualitative physics of the result. In the following, the initial phase (θ_0) of the motion is taken to be 0 (by shifting the origin of time).

$$x(t) \approx A\cos(\omega t) - \frac{\alpha A^2}{2\omega_0^2} + \frac{\alpha A^2}{6\omega_0^2}\cos(2\omega t)$$

$$\omega \approx \omega_0 + \left(\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3}\right) A^2$$

Characteristics of non-linear oscillators

Here are some essential notable things about this solution.

- **New effective equilibrium position** The time averaged value of x is not zero any more, but $-\frac{\alpha A^2}{2\omega_0^2}$. And the offset is amplitude dependent!
- **Overtones/Higher-harmonics** Notice that $x(t)$ has frequency 2ω. At higher orders of perturbation, further multiples of ω will appear.
- **Frequency softening/hardening** The angular frequency ω is now different from ω_0. Assuming $\alpha, \beta > 0$, we see that β increases the frequency (hardening), and α decreases it (softening).

[3]Interested readers should read Landau § 28. The perturbation method is slightly more complicated than the usual kind.

How do we understand these characteristics?

- **New effective equilibrium position** Assuming $\alpha > 0$, it is offset to the negative side. Why? Because that is where the leading order potential energy correction $\propto \alpha x^3$ is negative. So, while $x = 0$ is still the equilibrium point in a mathematical sense, it is meaningless to call that the equilibrium position. The more meaningful equilibrium position is the time average value of x, which is the observable of physics. In terms of this effective equilibrium position, the particle is no longer at 0. Instead, the particle is shifted towards a more attractive part, which is the $x < 0$ region, and thus the effectively speaking the mean position of x shifts more and more towards left as the amplitude of the oscillation A becomes larger and larger. This is precisely why when materials are heated up and all those molecules inside them make more vibrations they generally expand (because the potential between molecules is hard – steep slope – on short distance and soft – gentle slope – on large distance).

- **Overtones/Higher-harmonics** That the overtones would occur in non-linear oscillators is not that surprising from the perturbation point of view. The perturbation terms like $-m\alpha x^2$ or higher terms will give the driving force of $\cos^2(\omega t)$ and higher powers. They create sinusoidal functions with angular frequency 2ω and higher multiples of ω. In non-linear optics, this type of phenomenon is extremely important. Typically a non-linear crystal is illuminated with intense laser light. Due to the non-linear effect, higher frequency light comes out. Light of frequency as high as $O(100)$ times that of the original light is generated by repeating such process.

- **Frequency softening/hardening** As non-linear terms in the potential modify the overall shape of the potential, it is not surprising that the natural frequency changes as well. The above result shows that, for small perturbation, the cubic term softens the frequency, i.e. decreases the frequency, while the quartic term hardens the frequency.

19.3 Resonance for non-linear systems

The resonance phenomena for linear systems (SHOs) get modified with interesting results for non-linear systems.

19.3.1 Amplitude jumps and hysteresis

For a linear system, we derived that

$$D = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_N^2)^2 + 4\beta^2\omega^2}}$$

where ω is the driving frequency and ω_N is the natural frequency (formerly known as ω_0).

In the last lecture, we learned that when non-linear potentials are treated perturbatively, then the natural frequency changes according to

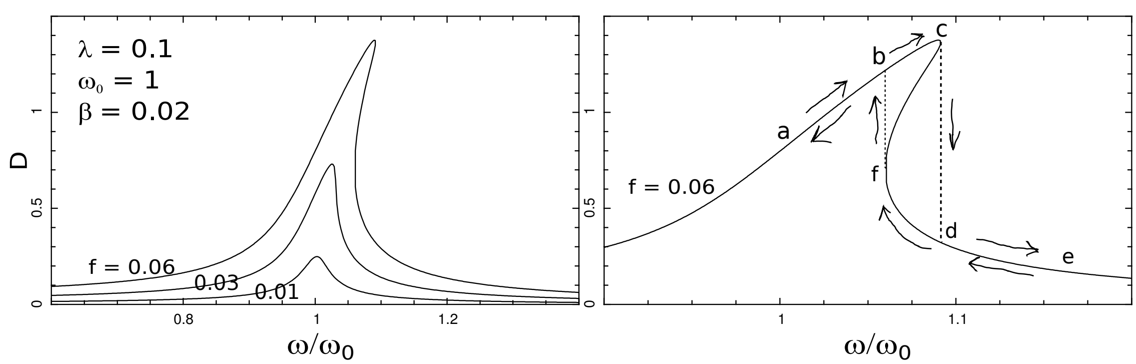
$$\omega_0'^2 \approx \omega_0^2 + \lambda D^2$$

Here, λ is a small number that occurs due to the cubic (softening) potential and the quartic (hardening) potential and so on.

This ω_0' is what we need to plug in for ω_N above, as that is the natural frequency of the system in the presence of the non-linear effects. So, for a non-linear system, we have the following equation

$$D = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2 - \lambda D^2)^2 + 4\beta^2\omega^2}}$$

This is now a cubic equation for D^2, and so it generally can have three roots for D! Here are some diagrams that are calculated as a function of $f = F_0/m$.



As expected the resonance frequency increases as f, and thus D, increases. However, a qualitatively new feature appears. The resonance profile is no longer single valued near the peak, if the driving force is sufficiently large! What this means is the following. As the value of ω is ramped up, the system will respond along the a-b-c line, at which point suddenly the amplitude must jump down to d, and then follow the path d-e. If the frequency is ramped down from high frequency, then the amplitude of the oscillator follows the path e-d-f-b-a, where f-b is a sudden jump of the amplitude. This is the phenomenon of amplitude jumps and hysteresis. As the phase shift is given by

$$\delta \; = \; \tan^{-1}\left(\frac{2\beta}{\omega_N^2 - \omega^2}\right)$$

it will also show jumps and hysteresis as D, since $\omega_N^2 \; \approx \; \omega_0^2 + \lambda D^2$ (see Figure 4.12 of the textbook).

19.3.2 Super-harmonic and sub-harmonic resonance

The above discussion is not all the story, of course. In the discussion above, we have tacitly assumed that the system acts as a linear oscillator except for the amplitude dependent natural frequency. But, we know that a higher harmonic generation is the characteristics of the non-linear operator. So, in a nonlinear system, the resonance response does not only appear at the natural frequency but also at higher harmonics of it. This has two implications. First, for a given fundamental resonance at $\approx \omega_N$, higher harmonic responses will be co-generated. This is the basis of the higher harmonic generation in non-linear optics. Second, there will be resonance lines at low frequencies, $\approx \; \omega_N/n$. This is the so-called **super-harmonic resonance**. What happens is that when a driving force with frequency $\omega = \omega_N/n$ is used, then due to the non-linearity, higher harmonics of ω will be induced. Eventually, the higher harmonic generated will match the natural frequency of the system ω_N, which induces a resonance response.

The **sub-harmonic resonance** is the opposite concept, in which the driving force with frequency ω produces a response at frequencies ω/n, and thus the resonance occurs at integer multiples of the fundamental frequency ω_N. How is this possible? Numerically, this effect starts to show when the system is driven strongly so that the amplitude response is large, and thus the non-linear effect is no longer small. So, it is plausible to qualitatively consider the origin of this effect, as a perturbation at the opposite limit, i.e. the perturbation theory when the non-linear effect dominates. To be very specific let us assume, by considering the cubic non-linear potential, that

$$\alpha x^2 + f(t) \; = \; \ddot{x} + 2\beta\dot{x} + \omega_N^2 x$$

and $|\alpha| D^2 \gg D$ (where D is the amplitude scale) so that the RHS can be considered as a perturbation. Such a large amplitude situation is appropriate for systems close to or in a chaotic regime. With $f(t) \; \propto \; e^{i\omega t}$, $x(t) \; \propto \; e^{i\omega t/2}$ in the zero-th order approximation. Plugging in this solution to the RHS to get the next order solution is equivalent to adding an external force term with frequency $\omega/2$. This would lead to the first order solution, which would then be governed by two frequencies: ω and $\omega/2$. Then, the second order solution will have three frequencies ω, $\omega/2$ and $\omega/4$. One can see that this process can continue indefinitely. Had the quartic and higher terms of the potential energy been added, then we would also have other frequencies such as $\omega/3$.

It can be seen that the sub-harmonic generation of an original frequency "input" to the system (ω), will lead to a longer and longer period (a cascade of period doubling or tripling), as ω become fractionalized (ω/n). Eventually, the system may become completely non-periodic and chaotic. The super-harmonic generation will not do such a thing, as the overall period remains the same in that case, when ω becomes doubled or tripled or more. Both effects (sub-harmonic and super-harmonic effects) will, of course, grow as the non-linearity of the system becomes more important.

19.4 Plane pendulum – fixed point, separatrix

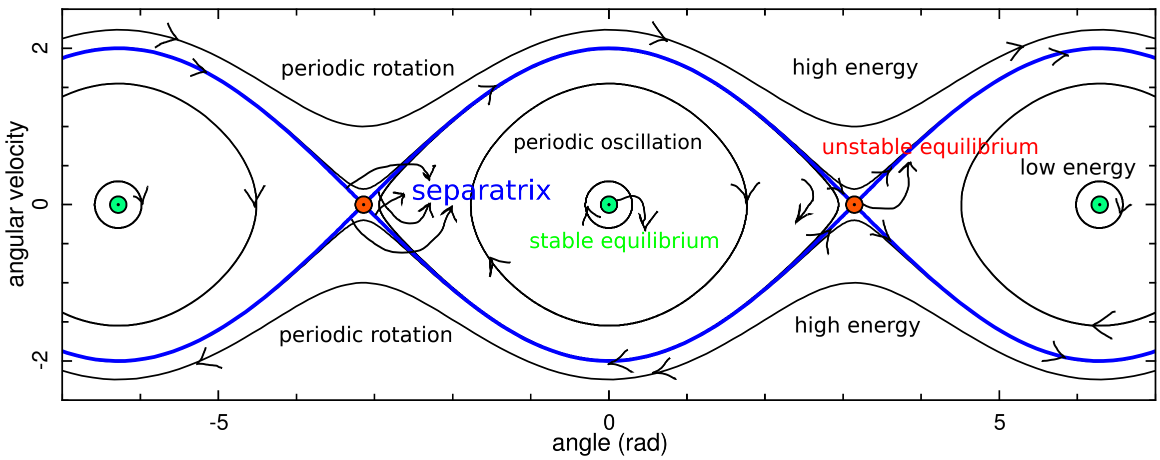
A plane pendulum is one of the simplest non-linear systems. It is *not* a chaotic system at all, but it does have an interesting feature to note: separatrix. Also, the notion of the fixed point can be learned.

Below are some examples of the motions represented in the phase space. First, we note that $\theta = 0$ modulo 2π is a stable equilibrium point (green points) and $\theta = \pi$ modulo 2π is an unstable equilibrium point (red points). Mathematically, the equilibrium points (stable or otherwise) are called "**fixed points**," as the system will stay there forever if found there. Physically, of course, this statement ("will stay there forever") is true only for the stable equilibrium point, as any "noises" or "fluctuations," thermal or otherwise, found in real systems will drive the system away from the unstable equilibrium point.

Around the stable equilibrium point, the motion is represented by a circle if ω_0 is taken to be 1 (close to the equilibrium point; this is the SHM) or a closed path (not so close to the equilibrium point). These correspond to $E < mgl$, assuming that we take U to be $-mgl\cos\theta$.

If $E > mgl$, then the motion corresponds to a rotation.

When $E = mgl$, then the phase diagram of the motion passes through the unstable equilibrium point. The diagram (blue curves) is an example of the so-called "**separatrix**." A separatrix is a phase space diagram that separates different types of motions, and it always passes through unstable equilibrium point(s). In the current example, any point found inside the separatrix corresponds to an oscillation, while any point found outside it corresponds to a rotation, while both kinds of motion are periodic.



On the separatrix, the motion is peculiar. If the system is found on the separatrix but not on the unstable equilibrium points, then the system is either approaching the unstable equilibrium point (like one of the arrows moving towards the red point) or departing it (like one of the two arrows moving away from the red point). The approach takes an infinite amount of time. The departure, on the other hand, will take a finite amount of time. However, the eventual fate of this departing motion is another ever-lasting approach to the unstable point. The departure and the approach are time-reversed states of each other.

19.5 Periodic motion to non-periodic motion

We have sketched above how a system might become non-periodic (sub-harmonic generation). However, this is still an active area of research and no general analytical method is known for predicting whether a given system will act chaotically. Instead, numerical methods play a central role in studies of non-linear systems. However, it is possible to identify an important category of simple motions that will *not* show chaotic behaviors.

We learned in a previous lecture that **all 1D bound motions, governed by a potential function $U(x)$, are periodic**. Clearly, a periodic motion is not chaotic. So, let us think about variations of this statement.

Here, we consider a potential $U(x)$ with a finite number of equilibrium points in any finite interval of x. Also, what we are interested in is the **long-term behavior** ($t \to \infty$), not the short-term behavior (transient behavior near the initial time).

- What if a dissipative force is added?
  If the force is dissipative, then the system would lose its energy over time. Eventually, it will be driven to a stable equilibrium point. So, a long term behavior is a trivial **fixed point** behavior. There is no motion in the long run. You might say that this an extremely boring, and non-chaotic "motion."

- What if a sinusoidal external driving force is added as well as a dissipative force, while the system is **linear**?
  This is the driven SHO problem that we already solved. In this case, a nice steady state solution with the same periodicity as the driving force emerges. So, a **periodic motion** is the long term behavior.

- What if a sinusoidal external driving force is added as well as a dissipative force, while the system is **non-linear**?
  We haven't dealt with any problem of this kind yet. In general, it is hard to know what will happen, and indeed a **chaotic behavior** can occur. And, this is not surprising given our arguments (sub-harmonic generation) above.

- How about in higher dimensions?
  Indeed. In higher dimensions, a mechanical problem, even without the dissipation and the external driving force, is generally very hard to solve. A **chaotic behavior** can occur at large amplitudes when the non-linearity becomes important! An example is a double pendulum, which has two degrees of freedom instead of one. More precisely speaking, what is important is that the problem of the double pendulum is a three dimensional problem in phase space (four to start with, energy conservation reduces the dimension by 1), higher than two (see the box below for the Poincaré-Bendixon theorem). Another example is the famous three body problem, which begot the concept of chaos in the first place (through the famed work of Poincaré).

- What if there is a dissipative force and an "anti-dissipative" force?
  An anti-dissipative force is one that adds energy, instead of subtracting it. The van-der Pol equation is one such example.

  $$\ddot{x} + \mu(x^2 - 1)\dot{x} + x \;=\; 0, \qquad \mu > 0$$

  This model has a self-regulating dissipation and "anti-dissipation" mechanism so that eventually any solution will converge to a periodic bound motion, which is called a **limit cycle**. There is obviously a trivial fixed point at $x = 0$ and $\dot{x} = 0$. These possible behaviors can be understood within the **"Poincare-Bendixon" theorem**. See the box below.

Poincaré-Bendixon theorem

Consider a differential equation $d\vec{r}/dt = f(\vec{r})$ where $\vec{r}$ is the position vector in a 2D plane,[a] $\vec{r} = (x, y)$. This theorem[b] states that the bound state solution of this equation converges to either an equilibrium point, a limit cycle or a separatrix. This means no chaotic behavior for a one dimensional closed mechanical system (closed in the sense that there is no external force $f(t)$ or any explicit time dependence on the force; dissipative forces are still allowed). Note that this theorem does not rule out a chaotic behavior of discrete dynamical systems (so-called "maps" such as the logistic map, Arnold's map, and the baker's map), nor does it rule out a chaotic behavior if the system is driven by an external force (so that f acquires an explicit time dependence; driven simple harmonic oscillator and driven van der Pol equation are examples of chaotic systems).

[a] or a cylinder, or a sphere, etc.

[b] Introduction to Applied Nonlinear Dynamical Systems and Chaos Texts in Applied Mathematics, 2003, Volume 2, 117-121

Limit cycle

Consider a 2D phase space. If there is a closed trajectory in the phase space to which other trajectory converges to as $t \to \infty$ or $-\infty$ is called a limit cycle. This behavior is exhibited in some non-linear systems. The van der Pol oscillator is a good example of such a system.

19.6 Chaos

In the discussion so far, we have identified some main features of nonlinear systems: subharmonic and superharmonic resonances, amplitude jump and hysteresis.

One main concept that emerges from studies of nonlinear systems is the concept of chaos. In every day language, chaotic systems are unpredictable systems – like the weather system, a sports game, the cloud formation, or elements of life(!). Before we define exactly what chaos is, we can talk about what chaos is not. For instance, periodic motion is not chaotic. In considering the periodicity and the loss of periodicity and a possible emergence of a chaotic behavior, the Poincare-Dixon theorem sheds some light. It says that if the dimension of the phase space is either less than 3 and if there is no external driving force, then the motion can never be chaotic. For instance, a simple pendulum can never be chaotic, without an external force. Also, a two body problem like the earth pulled by the sun is never chaotic, as, the effective phase space dimension in this case is just one. On the other hand, the phase space dimension of a three body problem or a double pendulum is greater than 2, and chaotic motions do occur for such a problem.

A chaotic system can be defined as a system showing an extreme sensitivity to the initial condition. The rigorous definition is often given in terms of the **Lyapunov exponent**. It is defined as λ in

$$\delta Z \approx e^{\lambda t} \varepsilon$$

as $t \to \infty$ and $\varepsilon \to 0$, where ε is the initial separation of the two initial conditions, and δZ is the final separation ($t \to \infty$) of the two initial conditions. If $\lambda > 0$, then the exponential factor diverges and so δZ will be finite, if we wait long enough, for any infinitesimal ε.

Note that in the above equation, in order to get $\delta Z \sim 1$, the waiting time $t \sim -\frac{1}{\lambda} \ln \varepsilon$ is necessary. So, even when ε is a small number, the waiting time will be modest, if $1/\lambda$ is of the order of the inherent period of the system (typically the case). By the same token, even when ε is reduced by a huge factor such as 10^4, the increase of the waiting time for observing the chaotic behavior is only mild, due to the logarithmic ratio 4 (detailed calculations left for students' work! – some calculator work seems necessary to go beyond a crude estimate).

A chaotic motion is a bound motion with a positive Lyapunov exponent. A bound motion is defined as a motion confined to a finite region in the phase space. For linear systems, $\lambda \leq 0$ for a bound motion (see Section **??**). For nonlinear systems, $\lambda > 0$ *can* occur for a bound motion, leading to a chaotic motion.

The above definition of chaos means that no matter how close two initial conditions are in the phase space, sooner or later the two motions will be completely separated in the phase space with a finite distance between them. [This is quite different from the behavior of a linear system, where the two nearly identical initial conditions will result in two nearly identical motions at *any time*. This was discussed in Lecture Note 5.] And, they are non-periodic motions. And, they are bound motions... If it seems a bit hard to imagine what kind of motion that is, then it is probably because it is hard to imagine. What happens in chaotic motions is repeated "stretching" and "folding" that remaps points of a bound region to the same region. This results in very complicated self-similar dense structures. Such a structure is called a fractal.

Science of . . . chaos?

This may sound a bit of an oxymoron. How can there be a science of chaos, if, by definition, a chaos means a hopelessly unpredictable situation.
Oddly enough, in modern physics, terms like "chaos," "frustration," "disorder," "complexity" and "symmetry breaking" have come to assume plenty of importance, although when you first hear them they might make you frown. Indeed, we are learning, in various contexts, that something novel, something genuinely creative, tends to happen in the vicinity of "chaos" or a genuinely unpredictable situation. They may include interesting pattern formations due to self-organization (how do clouds form?), or fundamental physics associated with symmetry breaking (the superconductivity and the Higgs boson physics).

Poincaré section

A chaotic motion traces a path in the phase space in a very complicated way. Consider a motion of a single particle in one dimension. Then the phase space is two dimensional, and including time, we have a three dimensional space to consider. As the phase path progresses in the phase space, the result can be a very complicated one. Poincaré noted that if a discrete set of times are sampled "wisely," then the resulting map can make it possible to distinguish between periodic motions that could admit analytic solutions, or chaotic motions that do not. There is no general method known to construct a Poincaré section, though, and a problem specific method is applied, although there exist methods developed for certain classes of problems. For a driven non-linear 1D oscillator, the Poincaré section can be easily defined as a "stroboscope" picture in phase space, taken with the period of the driving force. In a chaotic regime, a fractal pattern characterizes the Poincaré section (Figure 4-19 of the book).

Attractor

A set of point(s) to which the motion converges for a dissipative system. This can be a fixed point or a limit cycle (non-chaotic motion), or a strange/chaotic attractor (chaotic motion). Strange attractors are fractals. An example of a strange attractor is the Lorenz attractor, governed by the following equation.

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

19.6.1 Maps

A dynamical system can be defined as a discrete map mathematically. In this case, the dynamics is given as a series: $x_0 \to x_1 \to x_2 \to \cdots \to x_n \to \cdots$.

The Lyapunov exponent, λ, for the discrete map is defined as

$$d_n = \exp(\lambda n)\varepsilon$$

in analogy with the continuous case in the previous section. Here $d_n = |x_{1,n} - x_{2,n}|$ and $\varepsilon = |x_{1,0} - x_{2,0}|$, where the subscripts 1 and 2 mean two instances of the map with minutely different initial conditions, $\varepsilon \to 0$. It is left for readers' work to show that (e.g. see page 176 of the textbook):

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln \left| \frac{df(x_n)}{dx_n} \right|$$

An example of a map is a logistic map.

$$x_{n+1} = \alpha x_n(1 - x_n)$$

Here, $0 \le x_n \le 1$, $n = 0, 1, 2, \cdots$.

First of all, for discrete systems, the Poincare-Bendixon theorem does not apply, and this type of one-dimensional map *can* shows chaotic behavior – and it certainly does!

The logistic map can be a crude model of a biological system in an isolated environment with finite resources. If there is a small population to begin with, then the system will start reproducing and grow (assuming $\alpha > 1$), as the above equation behaves as $x_{n+1} \approx \alpha x_n$ if $x_n \ll 1$. However, if the population is too large, then it will consume all finite resources and the population will collapse. This is the behavior at the other end $x_n \to 1$, where $x_{n+1} \approx \alpha(1 - x_n)$. n may be thought of as some unit of time, like a year. This model has an interesting behavior. First of all, an obvious solution does emerge. A stable and steady population that represents the equilibrium between the finite resources and the population's tendency to reproduce. However, as the parameter α is tweaked up, more interesting behaviors appear. A sort of "bi-stable" situation occurs, where the population alternates between two values year by year! This is called the "bifurcation." It is analogous to the periodicity doubling in non-linear oscillators that we discussed. Each of those bifurcated points can bifurcate again, which means the appearance of period 4. This process can continue, and pretty soon, a chaotic behavior emerges.

Figures T4.22, T4.23, and T4.25 of the textbook are to be understood, in the context with these descriptions.

Notice that in Figure T4.25, the Lyapunov exponent becomes 0 whenever the bifurcation occurs. It is as though whenever the bifurcation occurs, the system is "knocking on the door of the chaos." As the bifurcation process continues, the system finally becomes chaotic (the Lyapunov exponent becoming greater than 0) as the α value becomes greater than $\approx$ 3.57. However, when α value increases more, the system shows a return to non-chaotic behavior in some finite value ranges of α.

An interesting characteristic of a chaotic map such as the logistic map is the so-called "Feigenbaum's number," which is defined as $\lim_{n\to\infty} \delta_n = \Delta\alpha_n/\Delta\alpha_{n+1}$, where $\Delta\alpha_n = \alpha_n - \alpha_{n-1}$, and α_n is the value of α for the n-th bifurcation point [Example T4.2 of the book].