Notes for Lecture 17

Coupled Oscillators, cont.

You should read the previous lecture note carefully, if you have not.

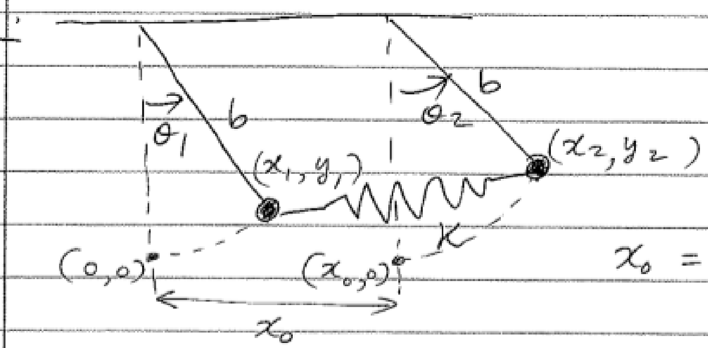
Essentially, the problem of coupled oscillators is to reduce a problem that seems extremely complicated to a problem that seems extremely simple – a system of independent normal modes. This is quite a beautiful and powerful thing to accomplish. The formal theory was presented in the last lecture.

One word of note – this process of reduction is valid whether we are dealing with a classical mechanical system or a quantum mechanical system. In quantum mechanical systems, we "simply" need to quantize the problem of each normal mode, which you will learn how to in a quantum mechanics class. Indeed, what you learn in coupled oscillators will be of immense importance in higher level courses (quantum mechanics, statistical mechanics, solid state physics, quantum field theory, etc).

Another word of note – the normal mode is not limited to vibrational modes. Translational and rotational modes are important normal modes.

In this note, we look at another example. Note that this example is essentially identical with the example of the last lecture, in terms of the mathematics involved. Here, however, at the end, we do the problem slightly differently (in terms of the choice of the eigenvector scale). Also, this examples is a somewhat difficult problem to set up, as we first need to reduce the Lagrangian to a quadratic form using approximations. Please make sure you understand all little details clearly!

Ex.



x_0 = natural length of spring

$$(x_1, y_1) = (b\, s_{\theta_1},\ b\,(1 - c_{\theta_1}))$$

$$(x_2, y_2) = (x_0 + b\, s_{\theta_2},\ b\,(1 - c_{\theta_2}))$$

distance between two masses

$$= \sqrt{\{x_0 + b(s_{\theta_2} - s_{\theta_1})\}^2 + \{b(c_{\theta_1} - c_{\theta_2})\}^2}$$

$$= \sqrt{x_0^2 + 2bx_0(s_{\theta_2} - s_{\theta_1}) + b^2(s_{\theta_2} - s_{\theta_1})^2 + b^2(c_{\theta_1} - \theta_2)^2}$$

$$\approx x_0 \left\{ 1 + \frac{b}{x_0}(s_{\theta_2} - s_{\theta_1}) \right\}$$

we need to keep terms up to $x_0 + \left(\begin{matrix}\text{linear in}\\ \theta_1, \theta_2\end{matrix}\right)$

Δx of the spring

$$= b(s_{\theta_2} - s_{\theta_1})$$

Δx is linear in (θ_1, θ_2)
$\therefore U \propto$ quadratic.

$$U = \frac{1}{2} k\, b^2 (s_{\theta_2} - s_{\theta_1})^2 + mgb(1 - c_{\theta_1}) + mgb(1 - c_{\theta_2})$$

$$\approx \frac{1}{2} k b^2 (\theta_2 - \theta_1)^2 + \frac{1}{2} mgb(\theta_1^2 + \theta_2^2)$$

$$T = \frac{1}{2} m\, b^2 (\dot\theta_1^2 + \dot\theta_2^2)$$

$$\overleftrightarrow{M} = \begin{bmatrix} b^2 m & 0 \\ 0 & b^2 m \end{bmatrix} \qquad \overleftrightarrow{A} = \begin{bmatrix} mgb + kb^2 & -kb^2 \\ -kb^2 & mgb + kb^2 \end{bmatrix}$$

$$\overleftrightarrow{A}\, \vec{T_i} = \omega_i^2\, \overleftrightarrow{M}\, \vec{T_i}$$

$$\left| \overleftrightarrow{A} - \omega^2 \overleftrightarrow{M} \right| = \begin{vmatrix} mgb + kb^2 - \omega^2 b^2 m & -kb^2 \\ -kb^2 & mgb + kb^2 - \omega^2 b^2 m \end{vmatrix}$$

$$= 0$$

$$(mg + kb - \omega^2 bm)^2 = (kb)^2$$

$$\omega^2 bm = mg + kb \pm kb$$

$$= mg \quad \text{or} \quad mg + 2kb$$

$$\omega^2 = \frac{g}{b} \quad \text{or} \quad \frac{g}{b} + \frac{2k}{m}$$

(soft) (hard)

$$\omega_1^2 = \frac{g}{b} \Rightarrow \overleftrightarrow{A} - \omega^2 \overleftrightarrow{M} = \begin{bmatrix} kb^2 & -kb^2 \\ -kb^2 & kb^2 \end{bmatrix}$$

$$\therefore \vec{T_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ up to a multiplicative const.

$$\omega_2^2 = \frac{g}{b} + \frac{2k}{m} \Rightarrow \overleftrightarrow{A} - \omega^2 \overleftrightarrow{M} = \begin{bmatrix} -kb^2 & -kb^2 \\ -kb^2 & -kb^2 \end{bmatrix}$$

$$\therefore \vec{T_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ "

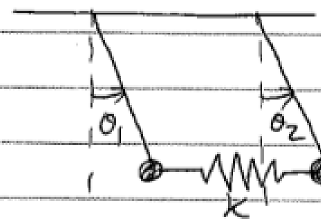
$\vec{T}_1 \eta_1$

For the soft mode

$$\omega_1^2 = \frac{g}{b} \Rightarrow \vec{\theta} = \overleftrightarrow{T} \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix}$$

$$\theta_1 = \theta_2 = \eta_1$$

motion for $\rightarrow$ η_1

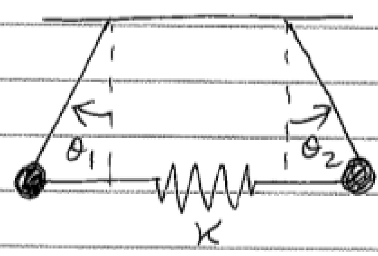


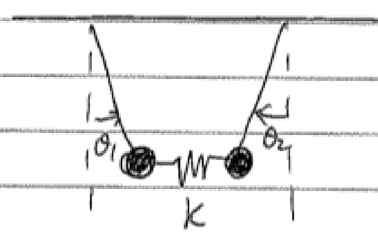
No spring force.

$\omega_1^2 = \frac{g}{b}$ $\leftarrow$ just like simple pendulum.

For the hard mode, $\omega_2^2 = \frac{g}{b} + 2\frac{k}{m}$

motion for $\rightarrow$ η_2





$$\vec{\theta} = \overleftrightarrow{T} \begin{bmatrix} 0 \\ \eta_2 \end{bmatrix} = \vec{T}_2 \eta_2$$

$$= \begin{bmatrix} \eta_2 \\ -\eta_2 \end{bmatrix}$$

$$\theta_1 = -\theta_2 = \eta_2$$

Note that in this case if g is "turned off" then $\omega_2^2 = \frac{2k}{m}$ ... why?

Internal motion of 2 body with potential $U = \frac{1}{2} k x^2$, reduced mass $= \frac{m}{2}$

$$\therefore \omega^2 = \frac{k}{\frac{m}{2}} = \frac{2k}{m}$$

To complete the solution, ask
what are η_1, η_2 ?

(normal coordinates)

$$\vec{x} = \overleftrightarrow{T}\vec{\eta} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \vec{T}_1 & \vec{T}_2 \end{array}$$

$$\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{\theta} \qquad \overleftrightarrow{T}^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore\ \eta_1 = \frac{1}{2}(\theta_1 + \theta_2), \quad \eta_2 = \frac{1}{2}(\theta_1 - \theta_2)$$

<u>Note</u>. If we defined $\overleftrightarrow{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

($\because$ a multiplicative constant at our disposal for <u>each</u> column vector)

then $\overleftrightarrow{T}$ is an orthogonal matrix!

$$\overleftrightarrow{T}^{-1} = \overleftrightarrow{T}^{t} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\vec{x} = \overleftrightarrow{T}\vec{\eta} \Rightarrow \begin{aligned} x_1 &= \frac{1}{\sqrt{2}}(\eta_1 + \eta_2) \\ x_2 &= \frac{1}{\sqrt{2}}(\eta_1 - \eta_2) \end{aligned}$$

$$\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{x} \Rightarrow \begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}}(\theta_1 + \theta_2) \\ \eta_2 &= \frac{1}{\sqrt{2}}(\theta_1 - \theta_2) \end{aligned}$$

$\overleftrightarrow{T}$ is <u>not</u> always orthogonal, though. (See next.)