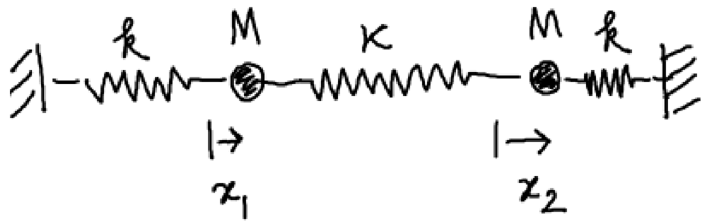
Notes for Lecture 16

Coupled Oscillators

We discuss one of the most important general topics of mechanics, the problem of coupled oscillators. We will approach this problem using a simple example, as defined below.



16.1 Newtonian EOM method

This method always works for any coupled oscillator problem. However, it may not be clear why it works. For that, see the Lagrangian method section.

16.1.1 Normal modes and their frequencies

For the above problem, the equation of motion for each mass is written, by inspection,

$$\begin{aligned} M\ddot{x}_1 &= -kx_1 - \kappa(x_1 - x_2) \\ M\ddot{x}_2 &= -kx_2 - \kappa(x_2 - x_1) \end{aligned}$$

This method starts out by assuming that a solution of the following form exists.

$$
\begin{aligned}
x_1 &= u \exp(i\omega t) \\
x_2 &= v \exp(i\omega t)
\end{aligned}
$$

This form of solution, where, all generalized coordinates oscillate at the same frequency is called a **normal mode** solution.

As in the SHM problem, here we decide to do solve the problem in the complex number domain, as we expressed x_1 and x_2 in terms of complex function $\exp(i\omega t)$. Likewise, u and v are complex numbers. We have in mind that after we are done obtaining the complete solutions of x_1 and x_2, we will take their real parts, which will be the real solutions.

By plugging in this normal mode solution, we get the following set of equations:

$$
\begin{aligned}
-M\omega^2 u &= -ku - \kappa(u - v) \\
-M\omega^2 v &= -kv - \kappa(v - u)
\end{aligned}
$$

Note that the $\exp(i\omega t)$ factor has been divided out from these equations, since it is a common factor for each term.

These two equations can be rewritten as

$$
\begin{aligned}
\overleftrightarrow{B}\vec{V} &= 0 \\
\overleftrightarrow{B} &\equiv \begin{pmatrix} k + \kappa - M\omega^2 & -\kappa \\ -\kappa & k + \kappa - M\omega^2 \end{pmatrix} \\
\vec{V} &\equiv \begin{pmatrix} u \\ v \end{pmatrix}
\end{aligned}
$$

We like to solve the above matrix equation for $\vec{V}$. Clearly, there is a trivial solution $\vec{V} = 0$. In fact, if $\overleftrightarrow{B}$ is invertible, then we see that $\overleftrightarrow{B}^{-1}\overleftrightarrow{B}\vec{V} = \vec{V} = \overleftrightarrow{B}^{-1}0 = 0$, i.e. $\vec{V} = 0$.

Therefore, we conclude that if $\overleftrightarrow{B}$ is invertible then we will get only the trivial solution, corresponding to no motion at all. This solution is definitely not what we want.

For a non-trivial solution, then, $\overleftrightarrow{B}$ must be singular (or non-invertible), namely $|\overleftrightarrow{B}| = 0$.

This gives the **secular equation** for ω^2:

$$(k + \kappa - M\omega^2)^2 - \kappa^2 \;=\; 0$$

From which we get

$$\omega \;=\; \sqrt{k/M}, \quad \sqrt{(k + 2\kappa)/M}$$

These are examples of the so-called **normal mode frequencies**. They represent the natural frequencies for a couple oscillator system.

So, using the assumption that a normal mode exists, we found two of them. The reason is that there are two degrees of freedom in this problem. This is very general. If there are n degrees of freedom in a coupled oscillator problem, then there are n normal modes.

16.1.2 Shape of normal modes

Having obtained the normal mode frequencies, we can ask what each normal mode looks like. For this we need to solve for $\vec{V}$.

For $\omega \;=\; \sqrt{k/M}$, we see that $\overleftrightarrow{B} \;=\; \left(\begin{smallmatrix} \kappa & -\kappa \\ -\kappa & \kappa \end{smallmatrix}\right)$. So, $\overleftrightarrow{B}\vec{V} \;=\; 0$ with $\vec{V} \;=\; \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ leads to the following condition:

$$u \;=\; v$$

This means that the two masses will move completely in sync, or **in phase**. We call this a **symmetric mode**, since the displacement on the left and the displacement on the right are identical.

It is easy to see that why this mode has the frequency $\sqrt{k/M}$. As the spring at the center is never stretched or compressed, it is as though it does not exist!

For the other mode, $\omega \;=\; \sqrt{(k + 2\kappa)/M}$, we see that $\overleftrightarrow{B} \;=\; \left(\begin{smallmatrix} -\kappa & -\kappa \\ -\kappa & -\kappa \end{smallmatrix}\right)$, which means that

$$u \;=\; -v$$

This means that the two masses move completely **out of phase**. We call this an **anti-symmetric mode**, since the displacement on the left is opposite in sign to the displacement on the right, while equal in magnitude. This type of mode is also called a breathing mode.

It is also easy to see why this mode has the frequency $\sqrt{(k + 2\kappa)/M}$. In this mode, when the spring on the left is compressed or extended by x_1, then the spring at the center is extended or compressed by twice as much. The net result is an effective spring constant $k + 2\kappa$ acting on the mass on the left. The same conclusion can be easily made for the mass on the right as well.

To prepare for the next discussion, let us write explicitly the amplitudes associted with each normal mode.

For the first normal mode, $x_1 = x_2 = u\exp(i\omega t)$. Let us take $u = D_1\exp(i\phi_1)$, with two real constants D_1 and ϕ_1. Finally, taking the real part, we can write

$$x_1 = x_2 = D_1\cos(\omega_1 t + \phi_1)$$

where $\omega_1 \equiv \sqrt{k/M}$.

For the second normal mode, $x_1 = -x_2 = u\exp(i\omega t)$. Let us take $u = D_2\exp(i\phi_2)$, with two real constants D_2 and ϕ_2. Taking the real part, we can write

$$x_1 = -x_2 = D_2\cos(\omega_2 t + \phi_2)$$

where $\omega_2 \equiv \sqrt{(k + 2\kappa)/M}$.

16.1.3 General solution

Let us note that the EOM that we started with has an important fact in common with the SHM EOM (LN 6). The EOM is linear. All terms are of linear order in x_1, x_2 or their time derivatives. This means that the two normal mode solutions that we found above, if simpled added (or formed into a linear combination in general), will give another valid solution.

Keeping in mind that this problem involves two amplitudes, we realize that a solution of this problem is a vector $\binom{x_1}{x_2}$. Above, we obtained two such solutions for the two normal modes. If we just add them up (which is all that needs to be considered in this case as each solution already includes an arbitrary multiplicative constants D_1 and D_2), then we get another solution $\binom{x_1}{x_2}$, as given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} D_1\cos(\omega_1 t + \phi_1) \\ D_1\cos(\omega_1 t + \phi_1) \end{pmatrix} + \begin{pmatrix} D_2\cos(\omega_2 t + \phi_2) \\ -D_2\cos(\omega_2 t + \phi_2) \end{pmatrix}$$

Thus, we obtain the following general solutions for this problem

$$\begin{aligned} x_1 &= D_1\cos(\omega_1 t + \phi_1) + D_2\cos(\omega_2 t + \phi_2) \\ x_2 &= D_1\cos(\omega_1 t + \phi_1) - D_2\cos(\omega_2 t + \phi_2) \end{aligned}$$

Why are these general solutions? They are good solutions and they contain four integration constants $(D_1, D_2, \phi_1, \phi_2)$. These four constants can be fixed if sufficient number of initial conditions are given.

16.2 Lagrangian method

The above method of solving Newtonian EOM works well. Even when the system involves damping, it works well. On the other hand, it does leave something to be desired as a *formalism*. For a general formalism, the Lagrangian method that we will discuss now is much more satisfying.

16.2.1 Mass tensor, stiffness tensor

In the Lagrangian method, we examine the kinetic energy and the potential energy

$$\begin{aligned} L &= T - U \\ T &= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 \\ U &= \frac{1}{2}kx_1^2 + \frac{1}{2}\kappa(x_1 - x_2)^2 + \frac{1}{2}kx_2^2 \end{aligned}$$

If we wrote down the Lagrangian EOM using this Lagrangian, we would obtain the precisely the same EOM as we solved above. This is not the purpose of this section. Instead, let us turn T and U in a more interesting form.

$$\begin{aligned} T &= \frac{1}{2}\dot{\vec{x}}^t \overleftrightarrow{M} \dot{\vec{x}} \\ U &= \frac{1}{2}\vec{x}^t \overleftrightarrow{A} \vec{x} \end{aligned}$$

Here, $\vec{x} \equiv \binom{x_1}{x_2}$. The matrices defined here are the **mass tensor** ($\overleftrightarrow{M}$) and the **stiffness tensor** ($\overleftrightarrow{A}$). Comparing the two forms of T, we see easily that

$$\overleftrightarrow{M} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$$

For $\overleftrightarrow{A}$, let us note that

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}\kappa(x_1^2 - 2x_1x_2 + x_2^2) + \frac{1}{2}kx_2^2$$

For constructing $\overleftrightarrow{A}$, we take its non-diagonal terms to be symmetric to find

$$\overleftrightarrow{A} = \begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix}$$

It is very important to note that (1) by construction $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$ are real symmetric matrices and (2) $\overleftrightarrow{M}$ is a positive definite matrix. The latter means that $\vec{v}^t \overleftrightarrow{M} \vec{v}$ is zero only if $\vec{v} = 0$ and positive for any non-zero vector $\vec{v}$.

16.2.2 General formalism

In the above example, the vector $\binom{x_1}{x_2}$ is what we were solving for.

In general, what we are solving for is a vector of generalized coordinates

$$\vec{q} \equiv \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

We define a general coupled oscillator problem (in the harmonic approximation) as

$$\begin{aligned} L &= T - U \\ T &= \frac{1}{2} \dot{\vec{q}}^t \, \overleftrightarrow{M} \, \dot{\vec{q}} \\ U &= \frac{1}{2} \vec{q}^t \, \overleftrightarrow{A} \, \vec{q} \end{aligned}$$

where $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$ are real symmetric matrices and $\overleftrightarrow{M}$ is positive definite.

Generally, any system of particles in a stable equilibrium state is expected to show this form of Lagrangian near the equilibrium. This is because expanding around the equilibrium point, taking $\vec{q}$ as relative to the equilibrium, there will be no term linear in $\vec{q}$ in U. Likewise, the kinetic energy term will be at least quadratic in generalized coordinates, as it arises from v^2. If the generalized coordinates are defined such that there are higher order terms in T, then, those terms can be ignored using small angle/displacement approximation.

As one can see, the above problem covers a lot of physical situations! The description "any system of particles in a stable equilibrium state" applies to a lot of situations. In particular, molecules and solids are prime examples of this problem. This generality makes this problem pretty important.

16.2.3 General solution

The general approach to solve the above problem is the following. Let us consider a non-singular coordinate transformation[1] $\overleftrightarrow{T}$

$$\vec{q} = \overleftrightarrow{T}\vec{\eta}$$
$$\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{q}$$

Upon the coordinate transformation, we get

$$T = \frac{1}{2}\dot{\vec{\eta}}^t\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T}\dot{\vec{\eta}}$$
$$U = \frac{1}{2}\vec{\eta}^t\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T}\vec{\eta}$$

What would be really nice is if the transformed mass tensor ($\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T}$) and the transformed stiffness tensor ($\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T}$) are both diagonal. Can this wish come true? We will now show that the answer is YES! Let us assume that the wish came true[2]:

$$\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T} = \overleftrightarrow{M}_d$$
$$\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{A}_d$$

Here,

$$\overleftrightarrow{M}_d \equiv \begin{pmatrix} m_1^* & 0 & \cdots & 0 \\ 0 & m_2^* & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & m_n^* \end{pmatrix}$$

where m_i^*'s are effective masses and

$$\overleftrightarrow{A}_d \equiv \begin{pmatrix} k_1^* & 0 & \cdots & 0 \\ 0 & k_2^* & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & k_n^* \end{pmatrix}$$

where k_i^*'s are effective spring constants.

[1]This transformation matrix is called $\overleftrightarrow{a}$ in the textbook.

[2]These two equations mean that $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$ are simultaneously diagonalized by the coordinate transformation $\overleftrightarrow{T}$. If $\overleftrightarrow{T}$ is an orthogonal matrix, then the above equations mean $\overleftrightarrow{M}\overleftrightarrow{T} = \overleftrightarrow{T}\overleftrightarrow{M}_d$ and $\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{T}\overleftrightarrow{A}_d$: in this case, the column vectors of $\overleftrightarrow{T}$ are simultaneous eigenvectors of $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$. $\overleftrightarrow{T}$ can be taken as orthogonal if $\overleftrightarrow{M}$ or $\overleftrightarrow{A}$ is a constant times the identity matrix. This is often the case, but not always, making $\overleftrightarrow{T}$ non-orthogonal in general. So, in general, it cannot be said that the column vectors that make up $\overleftrightarrow{T}$ are simultaneous eigenvectors of $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$.

If $\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T} = \overleftrightarrow{M}_d$ is true, then $\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T}\overleftrightarrow{M}_d^{-1} = 1$. Inserting this in front of $\overleftrightarrow{A}_d$ in the equation $\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{A}_d$, we get $\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T}\overleftrightarrow{M}_d^{-1}\overleftrightarrow{A}_d$. Multiply $\overleftrightarrow{T}^{t-1}$ from the left and we get

$$\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{M}\overleftrightarrow{T}\overleftrightarrow{\lambda}$$

where $\overleftrightarrow{\lambda} = \overleftrightarrow{M}_d^{-1}\overleftrightarrow{A}_d = \begin{pmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & \omega_n^2 \end{pmatrix}$ and $\omega_i^2 = k_i^*/m_i^*$.

This matrix equation can be re-written as the eigenvalue equation form

$$\overleftrightarrow{A}\vec{T}_i = \omega_i^2\overleftrightarrow{M}\vec{T}_i$$

where $\vec{T}_i$ is the i-th column vector of $\overleftrightarrow{T}$. This eigenvalue equation is the so-called **generalized eigenvalue equation**.

With $\overleftrightarrow{A}, \overleftrightarrow{M}$ symmetric and $\overleftrightarrow{M}$ positive definite[3], the above eigenvalue equation can be shown to be always solvable to give a non-singular matrix $\overleftrightarrow{T}$. Read more about this well-known problem in any linear algebra book or a technical book such as "Numerical Recipes in C."

Note that to complete the theory we still need to show that such $\overleftrightarrow{T}$ found does diagonalize $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$ simultaneously, as we prescribed above: $\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T} = \overleftrightarrow{M}_d$, $\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{A}_d$. I.e., so far we have merely shown that $\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{M}\overleftrightarrow{T}\overleftrightarrow{\lambda}$ is a necessary condition for the simultaneous diagonalization. Showing that it is a sufficient condition is required to complete the theory. The following note shows how this proof is done ($n = 2$ case for simplicity, but generalizable to any n). Note that part of this proof is the "orthogonality" of $\vec{T}_i$ vectors with different eigenvalues, *if* the mass tensor $\overleftrightarrow{M}$ is used as the metric tensor[4]. Also, note that this note assumes that eigenvalues (ω_i's) are all distinct (non-degenerate). If any of them happen to be the same (degenerate eigenvalues), one can always take eigenvectors to be orthogonal (see footnote 6).

[3] You may wonder whether $\overleftrightarrow{A}$ is also positive definite for motions around a stable equilibrium. This is actually not true. This is because, $\vec{q}^t\overleftrightarrow{A}\vec{q}$ can be zero for non-zero $\vec{q}$, corresponding to a translational mode. On another note, notice that our formalism here would work even if the equilibrium is a (partially) unstable one. In such a case, some k_i^* values will be negative.

[4] So, in general, they cannot be said to be orthogonal in the normal Cartesian space, unless $\overleftrightarrow{M}$ gives a Cartesian metric – a constant times the identity matrix

Question (Advanced; "orthogonality")

We started with

(i) $\overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} = \begin{bmatrix} m_1^* & 0 \\ 0 & m_2^* \end{bmatrix}$ $\quad \overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T} = \begin{bmatrix} k_1^* & 0 \\ 0 & k_2^* \end{bmatrix}$

and derived

(ii) $\overleftrightarrow{A} \overleftrightarrow{T} = \overleftrightarrow{M} \overleftrightarrow{T} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$ $\quad \omega_i^2 \equiv k_i^* / m_i^*$

Can we ensure that (ii) implies (i) ?

Yes! ① First, note $\vec{T}_i^t \overleftrightarrow{M} \vec{T}_i$ = positive due to the positive definite nature of $\overleftrightarrow{M}$.

② $\overleftrightarrow{A} \vec{T}_i = \omega_i^2 \overleftrightarrow{M} \vec{T}_i$

$\vec{T}_i^t \overleftrightarrow{A} = \omega_i^2 \vec{T}_i^t \overleftrightarrow{M}$

$\therefore \vec{T}_j^t \overleftrightarrow{A} \vec{T}_i = \omega_i^2 \vec{T}_j^t \overleftrightarrow{M} \vec{T}_i = \omega_j^2 \vec{T}_j^t \overleftrightarrow{M} \vec{T}_i$

$\therefore \vec{T}_j^t \overleftrightarrow{M} \vec{T}_i = 0$ if $\omega_i \neq \omega_j$

Combining ①, ② $\Rightarrow \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} = \begin{bmatrix} m_1^* & 0 \\ 0 & m_2^* \end{bmatrix}$

$m_1^*, m_2^* > 0$

Then $\overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T} = \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} m_1^* \omega_1^2 & 0 \\ 0 & m_2^* \omega_2^2 \end{bmatrix}$

QED.

Note that since $m_i^* > 0$ (due to $\overleftrightarrow{M}$ being positive definite), $\omega_i^2 \geq 0$ if $k_i^* \geq 0$. This would be the case for motions around a stable equilibrium point. We will consider this case only in this note[5].

Note that by finding the matrix $\overleftrightarrow{T}$ we reduce the Lagrangian into a sum of n separate Lagrangian for n normal modes

$$L = \sum_i \left(\frac{1}{2} m_i^* \dot{\eta}_i^2 - \frac{1}{2} k_i^* \eta_i^2 \right)$$

Thus, each normal mode has a simple harmonic motion solution (with $\omega_i = \sqrt{k_i^*/m_i^*}$):

$$\eta_i(t) = D_i \cos(\omega_i t + \phi_i)$$

To find $q_i(t)$, all we need to do is to use $\vec{q} = \overleftrightarrow{T}\vec{\eta}$.

$$q_i(t) = \sum_{i=1}^{n} T_{ij} \eta_j(t)$$

If $2n$ initial conditions are given, then all constants D_i and ϕ_i can be determined. Generally, initial conditions are given in terms of q_i and $\dot{q}_i$. These initial conditions can be converted to initial conditions for normal coordinates and their time derivatives, using $\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{q}$. Using the initial conditions for η_i and $\dot{\eta}_i$ thus determined, D_i and ϕ_i can be determined easily.

What we just proved should impress you. We just showed that, no matter how many particles we have and no matter how they interact with one another, the motion near the stable equilibrium is described by n independent normal modes, where n is the total degrees of freedom. This is indeed a very impressive general result. As you can imagine, this result is nicely utilized in many disciplines of physics, such as statistical mechanics, condensed matter physics, and others.

16.2.4 Back to the example

Note that, for the simple example that we started with,

$$\overleftrightarrow{A} = \begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix}$$

[5] But, see footnote 3.

and $\overleftrightarrow{M} = M\overleftrightarrow{1}$. For the generalized eigenvalue equation to have non-trivial solution, we require

$$|\overleftrightarrow{A} - \omega^2 \overleftrightarrow{M}| = 0$$

Note that this **secular equation**, if we plug in $\overleftrightarrow{A}$ and $\overleftrightarrow{M}$, becomes exactly the same secular equation that we solved within the Newtonian EOM method. As for the $\overleftrightarrow{T}$ matrix, we see that each column vector $\vec{T}_i$ is sort of equivalent to the vector $\vec{V}$ found in the Newtonian method. Indeed, they are the same eigenvectors that satisfy the same eigenvalue equation. Just as the $\vec{V}$ vector in the Newtonian formalism, there is an overall scale that is freely adjustable for each eigenvector $\vec{T}_i$ in the Lagrangian formalism. **However, there is an important difference between them.** Each $\vec{T}_i$ should be a *real* vector and it should be chosen *definitely* as a non-zero vector. Namely, each $\vec{T}_i$ should be chosen as one specific vector that represents a direction in the n dimensional vector space[6]. Often, it is a good idea to choose $\vec{T}_i$ as a unit vector, if that makes $\overleftrightarrow{T}$ an orthogonal matrix. Often, it is OK to choose $\vec{T}_i$'s to have nice numbers (all integers). It is your choice.

For this particular example that we started with, the following shows the full solution. The scales chosen for T_1 and T_2 are quite reasonable.

$$\vec{T}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega_1 = \sqrt{k/M}$$

$$\vec{T}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \omega_2 = \sqrt{(k+2\kappa)/M}$$

$$\overleftrightarrow{T} = \begin{pmatrix} \vec{T}_1 & \vec{T}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\vec{x} = \overleftrightarrow{T}\vec{\eta}$$

$$x_1 = \eta_1 + \eta_2 = D_1 \cos(\omega_1 t + \phi_1) + D_2 \cos(\omega_2 t + \phi_2)$$

$$x_2 = \eta_1 - \eta_2 = D_1 \cos(\omega_1 t + \phi_1) - D_2 \cos(\omega_2 t + \phi_2)$$

$$\overleftrightarrow{T}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{x}$$

$$\eta_1 = \frac{1}{2}(x_1 + x_2) = D_1 \cos(\omega_1 t + \phi_1)$$

$$\eta_2 = \frac{1}{2}(x_1 - x_2) = D_2 \cos(\omega_2 t + \phi_2)$$

Here, $\overleftrightarrow{T}$ is almost orthogonal. Had we divided each $\vec{T}_i$ by $\sqrt{2}$, another reasonable choice for scales, $\overleftrightarrow{T}$ would have become orthogonal.

[6] In some cases, the eigenvalue problem may yield a *degenerate* eigenvalue. For instance, a doubly degenerate eigenvalue corresponds to $\omega_i = \omega_j$ for a certain pair of i, j. In this case, any two linearly independent vectors in a plane specified by the eigenvalue equation can be chosen.

16.2.5 Summary

Grand Summary

Start with $T = \frac{1}{2} \dot{\vec{x}}^t \overleftrightarrow{M} \dot{\vec{x}}$

$U = \frac{1}{2} \vec{x}^t \overleftrightarrow{A} \vec{x}$

($\vec{x}$ can be $\vec{q}$ (generalized coord.))

Solve $\overleftrightarrow{A} \vec{T}_i = \omega_i^2 \overleftrightarrow{M} \vec{T}_i$ eigenvalue eq.

- Secular eq $\left| \overleftrightarrow{A} - \omega_i^2 \overleftrightarrow{M} \right| = 0$

  gives eigenvalues ω_i^2 ← (normal mode frequency)

  Plugging ω_i^2 back to the eigenvalue eq. one gets eigenvectors $\vec{T}_i$

- $\vec{T}_i$ gives the info on the i-th (normal mode), i.e, how the x_j's are related to each other, through

  $\vec{x} = \vec{T}_i \eta_i$ ~~if only η_i is excited.~~

- The (normal coordinates) are given as

  $\vec{\eta} = \overleftrightarrow{T}^{-1} \vec{x}$ $\overleftrightarrow{T} = [\vec{T}_1 \, \vec{T}_2 \cdots \vec{T}_n]$

  $n \times n$ matrix (n ← degrees of freedom)

- In general,

  $\vec{x} = \overleftrightarrow{T} \vec{\eta}$ $\eta_i = D_i \cos(\omega_i t + \phi_i)$ (↑ D_i, ϕ_i: integration constants)

n independent SHOs !! →

- $L = \sum_i \left(\frac{1}{2} \dot{\eta}_i^2 m_i^* - \frac{1}{2} \eta_i^2 m_i^* \omega_i^2 \right)$