Notes for Lecture 14

Kepler problem and many body problem

14.1 Elliptical orbits

14.1.1 Turning points and angular momentum conservation

Let us come back to the discussion of elliptical orbits, applicable to comets as well as planets. In an elliptical orbit,

$$r_{min} = \frac{\alpha}{1+\varepsilon},\ r_{max} = \frac{\alpha}{1-\varepsilon}$$

These are "turning points" for the r motion, or **apsides**. In particular, these are called **pericenter** (r_{min}), or **apocenter** (r_{max}). Think "apex" to remember which is which. For objects orbiting around the Earth, we talk about perigee and apogee, and for objects orbiting around the Sun, we talk about perihelion and aphelion. Note that **the speed is maximum (minimum) at r_{min} (r_{max})**. This follows directly from the energy conservation $E = T + U = \frac{1}{2}\mu v^2 - k/r$. Call them v_{min} and v_{max}. At the turning points, $\vec{v} \perp \vec{r}$, and so the angular momentum conservation means for apsides,

$$r_{min} v_{max} = r_{max} v_{min}$$

14.1.2 Kepler's third law

The total area of an ellipse is πab. As we saw, the areal velocity is constant **for any central force**, and is given by $dA/dt = l/2\mu$. This proves Kepler's second law (this is no news to us; we did this in the previous lecture). Now, the period τ for an elliptical motion is given by $\tau = \pi ab/(dA/dt) = 2\pi ab\mu/l$. As $b = \sqrt{\alpha}\sqrt{a} = l\sqrt{a}/\sqrt{\mu k}$, **we get the following Kepler's third law**

$$\tau^2 \;=\; \frac{4\pi^2\mu}{k}a^3$$

Note that this law should read generally thus: "the period squared is proportional to the linear dimension of the orbit cubed." For instance, if we had eliminated a instead of b, then we would have gotten $\tau^2 \propto b^3$ with a different proportionality constant. Or, one could say that the average radius of an elliptical orbit is proportional to $\tau^{2/3}$. However, as it is written, the above form, expressed in terms of a, has a certain beauty. Likewise, note that $E = -k/(2a)$ for an elliptical orbit (bottom of page 9 of LN 13). For the reason that will be explained soon, these relations are worth remembering.

14.1.3 Virial theorem

The "virial theorem" for power law forces is a powerful theorem that can be handy very often. It is useful for describing planetary motions. Furthermore, when classical mechanics must be replaced by quantum mechanics, this theorem survives: so the theorem remains valid for electron motion in an atom. Here, we will not be distracted by the derivation of the theorem (see page 279 of the textbook or google), but simply state the theorem:

$$\langle T\rangle \;=\; \frac{n}{2}\langle U\rangle \qquad \text{if } \; U \propto r^n$$

where $r \;=\; |\vec{r}|$. Here, the notation $\langle Q\rangle$ means the average of quantity Q over one period (if the motion is periodic) or over a long time (if the motion is bound, periodic or not). In other words,

$$\langle Q\rangle \;=\; \frac{1}{T}\int_0^T dt\, Q$$

where $T \;=\; \tau$ (the period of motion, if applicable) or $T \to \infty$ (for a bound motion).

Let us talk specifically about the motion under Hooke's law force ($n = 2$) or the motion under Newton's law of gravity ($n = -1$). The motion under Hooke's law force

is always a bound motion, and it turns out that it is always periodic. As we saw in the last lecture, the motion under Newton's law of gravity can be bound ($0 \leq \varepsilon < 1$) or unbound ($\varepsilon \geq 1$). For the discussion of the Virial theorem, we focus on the bound motion only. In this case, the motion is also a periodic motion, tracing an ellipse. So, we will use $\langle Q \rangle$ in the first meaning, the time average over a period, in the rest of this section, and use a subscript τ to make this meaning clear as in $\langle Q \rangle_\tau$.

For the Hooke's law force, we get $\langle T \rangle_\tau = \langle U \rangle_\tau$. We are already familiar with this result from a previous homework problem.

Let us consider the current Kepler problem, $n = -1$.

According to this theorem, we have, for any elliptical orbit,

$$2\langle T \rangle_\tau = -\langle U \rangle_\tau$$

Since the energy $E = T + U$ is constant, it then follows that

$$E = -\langle T \rangle_\tau = \frac{1}{2}\langle U \rangle_\tau$$

As, $E = -k/(2a)$, this means that $\langle U \rangle_\tau = -k/a$ and $\langle T \rangle_\tau = k/(2a)$.

How to remember, or quickly re-derive, stuff?

Suppose you need to remember the above Kepler's third law (and some other crucial stuff that we derived above). **You can do it, very easily!** Do you have to memorize it? There is no way *I* can memorize such a formula! But, wait, there is a clever way to "recall," without doing all the complicated stuff that we just did! Just remember this one first: **the uniform circular motion is your best friend...** For the circular motion, we have

$$\frac{\mu v^2}{r} = \frac{k}{r^2} \qquad \text{Centripetal force equation}$$

$$E = T + U = \frac{\mu v^2}{2} - \frac{k}{r} \qquad \text{Energy conservation}$$

where r, v, T, U are all *constants*. By expressing the centripetal force equation in terms of τ, by noting $v = r\omega = r2\pi/\tau$, we get

$$\tau^2 = \frac{4\pi^2\mu}{k} r^3 \qquad \textbf{Kepler's 3rd law for circular orbit}$$

By multiplying the first equation by r, we get

$$-2T = U \qquad \textbf{Virial theorem example}$$

$$E = -T = \frac{U}{2} = -\frac{k}{2r}$$

The good news is that the last three equations remain valid even for elliptical orbits, if we make the following substitutions: ($\langle \cdots \rangle_\tau$: average over a period)

$$r \to a,\ T \to \langle T \rangle_\tau,\ U \to \langle U \rangle_\tau$$

For both circular orbits and elliptical orbits, we have:

$$\tau^2 = \frac{4\pi^2\mu}{k} a^3$$

$$\langle U \rangle_\tau = -2\langle T \rangle_\tau$$

$$E = -\langle T \rangle_\tau = \frac{\langle U \rangle_\tau}{2} = -\frac{k}{2a}$$

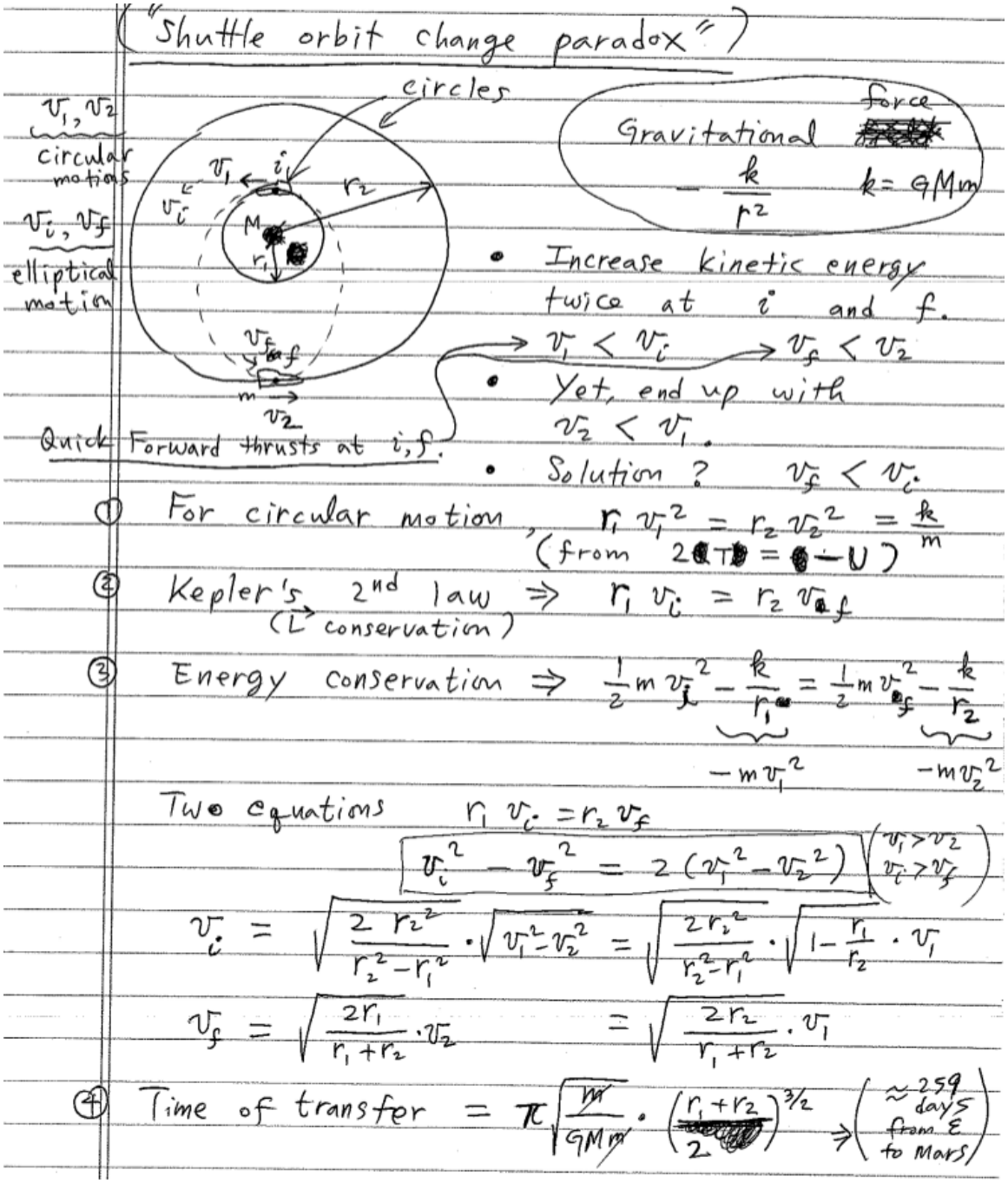
The angular momentum conservation, $l = \mu r^2\dot{\theta} = 2\mu dA/dt$ (valid for *any* central force) and these three equations are **crucial things to rather easily recall** as shown here (not memorize!).

14.1.4 Hohmann transfer

This transfer process can be shown to be the most fuel economic way for a space shuttle to go from one circular orbit to another circular orbit under the gravitational field of the Earth. The following page shows the summary of this transfer process, which consists of two quick forward thrusts. This whole process may sound a bit paradoxical at first. By accelerating twice, we go from a small circular path (with v_1) to a larger circular path (with v_2), but yet, we must have $v_2 < v_1$. The latter follows from the virial theorem as applied to the circular motion:

$$-U \;=\; 2T \;\text{ with }\; U \;=\; -\frac{k}{r},\; \text{and } T \;=\; \frac{1}{2}mv^2$$

This apparently paradox is easily resolved, however, since the speed decreases (from v_i to v_f) in the intermediate elliptical orbit. The following note shows how to compute the values of v_i and v_f knowing r_1, r_2 (and thus v_1 and v_2; these are easy to know since they are all related to circular motions). Note that we use none of complicated calculations here. All we use here are the virial theorem (step 1), the angular momentum conservation (step 2), energy conservation (step 3), and Kepler's third law (step 4), all of which must be easily recallable by you(!). Also, note that in this note, the reduced mass μ has been approximated to m (the mass of the shuttle) as we assume $M \gg m$, where M is the mass of the Earth.


"Shuttle orbit change paradox"
circles
v_1, v_2
circular motions
v_i, v_f
elliptical motion
v_1
i
r_2
v_i
M
r_1
v_f
f
m
v_2
Quick Forward thrusts at i, f.
Gravitational force
$-\frac{k}{r^2}$
$k = GMm$
• Increase kinetic energy twice at i and f.
$v_1 < v_i \rightarrow v_f < v_2$
• Yet, end up with $v_2 < v_1$.
• Solution ? $v_f < v_i$
① For circular motion, $r_1 v_1^2 = r_2 v_2^2 = \frac{k}{m}$ (from $2\langle T \rangle = -U$)
② Kepler's 2nd law ⇒ $r_1 v_i = r_2 v_f$ ($\vec{L}$ conservation)
③ Energy conservation ⇒ $\frac{1}{2} m v_i^2 - \frac{k}{r_1} = \frac{1}{2} m v_f^2 - \frac{k}{r_2}$
$-m v_1^2$ $-m v_2^2$
Two equations $r_1 v_i = r_2 v_f$
$v_i^2 - v_f^2 = 2(v_1^2 - v_2^2)$ ($v_1 > v_2$, $v_i > v_f$)
$v_i = \sqrt{\frac{2 r_2^2}{r_2^2 - r_1^2}} \cdot \sqrt{v_1^2 - v_2^2} = \sqrt{\frac{2 r_2^2}{r_2^2 - r_1^2}} \cdot \sqrt{1 - \frac{r_1}{r_2}} \cdot v_1$
$v_f = \sqrt{\frac{2 r_1}{r_1 + r_2}} \cdot v_2 = \sqrt{\frac{2 r_2}{r_1 + r_2}} \cdot v_1$
④ Time of transfer $= \pi \sqrt{\frac{m}{GMm}} \cdot \left(\frac{r_1 + r_2}{2}\right)^{3/2}$ ⇒ (≈ 259 days from E to Mars)

14.2 Many body system

Let us consider a system of particles, or a many body system.

Any classical mechanics object is a many body system by definition, as we noted in Lecture 1. So, here we make a sort of a full circle. Nevertheless, it is important to recognize certain key concepts such as the center of mass.

To start, let us assume that we have a system of particles, m_α, where α is the index for each particle, which we describe in an inertial frame. Let us define this inertial frame as our "LAB" frame. In this LAB frame, let $\vec{r}_\alpha$ be the position vector for m_α.

14.2.1 Center of mass

The center of mass coordinate $\vec{R}$ is defined as

$$\vec{R} \equiv \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{M}$$

where

$$M \equiv \sum_\alpha m_\alpha$$

is the total mass.

We now define the "CM" (center of mass) frame as the reference frame for which the origin is at $\vec{R}$. Note that the CM frame is generally *not* an inertial frame, as it is, in general, accelerating relative to the LAB frame, which we assumed to be an inertial frame.

Let the position vectors measured in the CM frame be $\vec{r}'_\alpha$.

By construction, then, we have $\vec{r}'_\alpha = \vec{r}_\alpha - \vec{R}$, or

$$\vec{r}_\alpha = \vec{R} + \vec{r}'_\alpha$$

Multiplying this equation by m_α, and then summing up over α, we get a trivial but pretty important result:

$$0 = \sum_\alpha m_\alpha \vec{r}'_\alpha$$

This is a trivial statement, since what it says is that the center of mass in the CM frame is zero. Of course, that follows how we defined the CM frame! Taking the time derivative, we get

$$\sum_\alpha m_\alpha \dot{\vec{r}}'_\alpha = 0$$

These two identities can be used to prove the following extremely useful kinematics relations.

$$\vec{P}_{tot} \stackrel{def}{=} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$
$$\vec{L}_{tot} \stackrel{def}{=} \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = M\vec{R}\times\dot{\vec{R}} + \sum_\alpha m_\alpha \vec{r}'_\alpha \times \dot{\vec{r}}'_\alpha$$
$$T_{tot} \stackrel{def}{=} \frac{1}{2}\sum_\alpha m_\alpha |\dot{\vec{r}}_\alpha|^2 = \frac{1}{2}M|\dot{\vec{R}}|^2 + \frac{1}{2}\sum_\alpha m_\alpha |\dot{\vec{r}}'_\alpha|^2$$

Proving these identities are left for your exercise (the first and the third done in class).

These identifies show the importance of the center of mass. All of these kinematical quantities are nicely split into two terms, the center of mass term with the mass M, and the internal term as measured in the CM frame. For the linear momentum the latter is zero by definition.

The above refer to the kinematics only. How about the dynamics?

$$\dot{\vec{P}}_{tot} = \vec{F}^{(e)} \stackrel{def}{=} \sum_\alpha \vec{F}^{(e)}_\alpha$$
$$\dot{\vec{L}}_{tot} = \vec{N}^{(e)} \stackrel{def}{=} \sum_\alpha \vec{r}_\alpha \times \vec{F}^{(e)}_\alpha$$

Here, $\vec{F}^{(e)}_\alpha$ is the total *external* force that applies to m_α. And, $\vec{N}^{(e)}$ is the total torque due to external forces. For sure, note that there are internal forces that apply to m_α as well, due to all other masses m_β with $\beta \neq \alpha$. However, due to the symmetry of space, the total momentum must be conserved for a closed system, and so is the total angular momentum. This is the reason why in these expressions only external forces appear, since without external forces $\dot{\vec{P}}_{tot} = 0$ and $\dot{\vec{L}}_{tot} = 0$.

Note that when these two equations are combined, one can conclude the following.

$$\begin{aligned}
\dot{\vec{L}}_M &= \frac{d}{dt}\left(M\vec{R}\times\dot{\vec{R}}\right) && \vec{L}_M \equiv M\vec{R}\times\dot{\vec{R}} \\
&= \vec{R}\times\vec{F}^{(e)} \\
\dot{\vec{L}}' &= \frac{d}{dt}\left(\sum_\alpha m_\alpha \vec{r}'_\alpha \times \dot{\vec{r}}'_\alpha\right) && \vec{L}' \equiv \sum_\alpha m_\alpha \vec{r}'_\alpha \times \dot{\vec{r}}'_\alpha \\
&= \vec{N}^{(e)\prime} \stackrel{def}{=} \sum_\alpha \vec{r}'_\alpha \times \vec{F}^{(e)}_\alpha
\end{aligned}$$

where $\vec{L}_{tot} = \vec{L}_M + \vec{L}'$, the sum over the angular momentum of the mass M at the center of mass coordinate and the angular momentum measured within the CM frame. $\vec{N}^{(e)\prime}$ is the torque around the center of mass due to all external forces.

Notice that the equation $\dot{\vec{L}}' = \vec{N}^{(e)\prime}$ is a highly non-trivial one, since the reference frame – the CM reference frame – is a non-inertial one. It is a testament to the special nature of the CM reference frame.

Lastly, the dynamical equation $\dot{\vec{P}}_{tot} = \vec{F}^{(e)}$ (and $\dot{\vec{L}}_M = \vec{R} \times \vec{F}^{(e)}$) can be seen as none other than Newton's second law for a "point particle," i.e., the equation from which we started all of these things!

14.2.2 Collisions

A collision generally refers to a process of a short lived process by which two objects come into contact and a violent exchange of forces.

The exchange of forces is conveniently quantified by the impulse.

$$\vec{I} = \int_{t_1}^{t_2} dt \vec{F}(t)$$

where $\vec{F}(t)$ is a force that is experienced by one object during the collision process. Putting $\Delta t \equiv t_2 - t_1$, we get

$$\vec{I} = \vec{F}_{ave}\, \Delta t$$

Usually, Δt is small. Even for a small period of time, the impulse is finite, due to a large force $\vec{F}_{ave}$ exchanged. For the object that receives the impulse, its momentum change during Δt is given by

$$\Delta \vec{P} = \vec{I}$$

Put another way, if Δt is short, and if $\vec{F}_{ave}$ is finite, then the impulse is small, and the effect of collision is negligible. This is the case of the "pulling the table cloth under the cups and plates" trick, sometimes played by waiters at a restaurant.

For collisions occurring in a closed system, the angular momentum and the momentum are always conserved. However, even for a closed system, a collision is generally inelastic – the initial total kinetic energy is not quite equal to the final total kinetic energy as some of the energy goes into modifying the internal state of matter (wear and tear). Often, however, a collision can be approximated well as an elastic collision. Even for a non-closed system, either the angular momentum or the momentum is often conserved along certain directions. In these cases, a close inspection of the system in terms of external torques or forces is required to make such a conclusion, while it is often the case that which quantity is conserved is rather obvious. However, as a general rule, if the collision between macroscopic objects is not said to be elastic, then it is likely not elastic – so one is advised *not* to assume an elastic collision without any good reason.