Notes for Lecture 13

Central force and the Kepler problem

Central force problems are important. All planet motion problems are of this kind. Also, crude but useful classical mechanics models of atoms and nucleons are of this kind. Crude, because classical mechanics fails for atoms and nucleons, but useful, because sometimes a good way to grasp the results of quantum mechanics is to start from classical mechanics and then to put in a quantization condition *by hand.*[1] Also, it should be noted that the fact that a symmetry leads to a conservation quantity is unchanged in quantum systems, and a very useful property such as the virial theorem continues to hold in quantum mechanics.

13.1 Two and many body problem, general

Here, we will consider a two body problem, with a more general form of potential than that for a central force problem.

Let us consider a two body problem with the potential that depends on the relative coordinate $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$.

$$L = \frac{1}{2}\left(m_1|\dot{\vec{r}}_1|^2 + m_2|\dot{\vec{r}}_2|^2\right) - U(\vec{r})$$

The total number of degrees of freedom is 6, 3 from $\vec{r}_1$ and 3 from $\vec{r}_2$. A different way

[1] This is the so-called "semi-classical picture" of quantum mechanics, due to Bohr and Sommerfeld. Such a picture was the foundation of the first-ever quantum theory of atoms, by Bohr, and is still useful in many cases.

to distribute the degrees of freedom is to define

$$\begin{aligned} \vec{R} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned}$$

Here, $M \equiv m_1 + m_2$, $\vec{R}$ is the **center of mass coordinate**, and $\vec{r}$ is the **relative coordinate**. The inverse transformation is, after some algebra,

$$\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{M}\vec{r} \end{aligned}$$

From this, one obtains

$$\begin{aligned} \dot{\vec{r}}_1 &= \dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}} \\ \dot{\vec{r}}_2 &= \dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}} \end{aligned}$$

Plugging this into the above Lagrangian, we get

$$L = \frac{1}{2}M|\dot{\vec{R}}|^2 + \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(\vec{r})$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the so-called **reduced mass**. This was a nice thing to do, since the two sets of degrees of freedom are now nicely separated. The **center of mass coordinate** degrees of freedom, $\vec{R}$, are governed by the free particle Lagrangian

$$L_{cm} = \frac{1}{2}M|\dot{\vec{R}}|^2$$

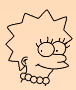
while the **relative coordinate**, or **internal**, degrees of freedom, $\vec{r}$, are governed by the dynamics of **one body with a reduced mass** μ under the influence of the interaction U:

$$L_i = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(\vec{r})$$

It is easily noted that L_{cm} is a highly symmetric Lagrangian. It is invariant both translationally and rotationally. So, the linear momentum of the center of mass, $M\dot{\vec{R}}$, and the angular momentum of the center of mass, $M\vec{R} \times \dot{\vec{R}}$, are conserved. Also, the energy, $\frac{1}{2}M|\dot{\vec{R}}|^2$, is conserved.

One can attribute the conservation of these quantities to the symmetry of space time. However, one must be mindful of the following fact. As we are talking now of a multiple-particle system, one has to extend the analysis of our lecture 9 ("symmetry and conservation") to a many particle system. In this more general case, the symmetry principle leads to the conservation of **the total momentum, the total angular momentum and the total energy**. As one can see rather easily (cf. homework), $M\dot{\vec{R}}$ is the total linear momentum, but $M\vec{R}\times\dot{\vec{R}}$ is not the total angular momentum and $\frac{1}{2}M|\dot{\vec{R}}|^2$ is not the total energy.

The following is valid for any many particle system, and should be noted with care.



Symmetry and conservation for many bodies

We consider an arbitrary *closed* system consisting of many constituent particles.
(1) The *total* momentum, the *total* angular momentum, and the *total* energy are conserved, by the homogeneity of space, the isotropy of space and the homogeneity of time, respectively.
(2) The motion of the center of mass is trivial: it is that of a free particle with the mass equal to the total mass of the system. The momentum of this motion is the total momentum of the system. The angular momentum and the energy associated with this motion are conserved, but they are not the total values of the system.
(3) The internal motions as described in the center of mass reference frame, has zero total momentum. However, they are characterized by generally non-zero total angular momentum and total energy, both of which are conserved. These properties follow from (1) and (2).

For (3) (and thus (1)) to come true, however, the form of the potential $U(\vec{r})$ must be restricted. Namely, it has to be of the "central force" form, by which it is meant that $U(\vec{r}) = U(r)$, i.e. the potential has to be isotropic. This is why we consider the central force.

By choice, here we are limiting ourselves to the case of a position dependent potential only. In general, however, one finds that the potential function can depend on the velocity as well (as would be the case if charged particles interact via Lorentz force). No matter what the form of the potential is, the potential function must be rotationally invariant.

Now, we turn to a particularly important problem: the "central force"

13.2 Two body with central force

Central force means that the potential, U, belongs in a simpler subclass of the class of simple potentials, $U = U(\vec{r})$, that we assumed for the most part of the previous section: U depends *only* on $r = |\vec{r}_1 - \vec{r}_2|$:

$$U \; = \; U(r)$$

This means that the force is only along the radial direction, and there is no angular dependence in the force.

$$\vec{F} \; = \; -\hat{r}\; dU/dr$$

From the discussion of the previous section, it should not come as a surprise at all that many fundamental forces of Nature are of this kind: Newton's gravitational law force, Coulomb force, screened Coulomb force, and Yukawa "force" (for nucleons).

From now on, we do not consider the motion of the center of mass, which is a trivial constant velocity motion. We consider the internal motion, only.

$$L \; = \; \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(r)$$

Let us note some basic things.

1. This Lagrangian is invariant rotationally, as it should be, as discussed above. This means that the angular momentum vector is conserved: $\vec{L} = \vec{r}\times\vec{p}$. As both $\vec{r}$ and $\vec{v} \propto d\vec{r}$ are perpendicular to the constant vector $\vec{L}$, it follows that $\vec{r}$ should remain in a plane. **It is a planar motion! The problem is effectively two dimensional!**

2. This Lagrangian, with no explicit time dependence, leads to the energy conservation, as it should, as discussed above. The energy is given by

$$E = \frac{1}{2}\mu|\dot{\vec{r}}|^2 + U(r)$$

3. This Lagrangian is apparently *not* translationally invariant on $\vec{r} \to \vec{r} + \delta\vec{r}$, if U is not a trivial constant. But, this fact is of little significance from the point of view of the symmetry of the overall problem. From the overall symmetry point of view, a translation operation means that the relative vector $\vec{r} = \vec{r}_1 - \vec{r}_2$ should be constant on translating $\vec{r}_1$ and $\vec{r}_2$. So, this Lagrangian does in fact have the translational symmetry from this overall symmetry point of view, as it should! And the system is always characterized by the zero total linear momentum, as $\partial L / \partial \dot{\vec{R}} = 0$, or because the system is "riding in the center of mass frame." The motion described by $\vec{r}$ is the relative motion of the constituent particles, and has no bearing on the overall translational symmetry or the total linear momentum.

13.2.1 Angular momentum conservation

For any central force, the angular momentum is conserved. As this means that the motion is effectively two dimensional, we can write the Lagrangian as

$$L \;=\; \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

in the **two dimensional** polar coordinate system (r, θ). The reason why we prefer the polar coordinate system to the Cartesian system is because the rotational invariance is easier to deal with in the polar system.

The canonical momentum for θ is given by

$$p_\theta \;=\; \partial L / \partial \dot{\theta} \;=\; \mu r^2 \dot{\theta}$$

This is, as we encountered a similar form a few times by now, the angular momentum associated with the θ rotation. This must be conserved, as L is rotationally invariant on $\theta \to \theta + \delta\theta$ for any constant $\delta\theta$. The Lagrangian EOM verifies that $(\partial L / \partial \theta = 0 = d(p_\theta)/dt)$. So, we have an **"integral of motion"** (cf. Lecture 5).

$$l \;\equiv\; p_\theta \;=\; \mu r^2 \dot{\theta}$$

where l is a constant.

Notice that $r^2 d\theta = |\vec{r} \times d\vec{r}|$ $(\because d\vec{r} = dr\hat{r} + rd\theta\hat{\theta}) = |\vec{r} \times (\vec{r} + d\vec{r})|$ $(\because \vec{r} \times \vec{r} = 0)$ is twice the area dA swept by the vector $\vec{r}$ during time dt, as $\vec{r}$ becomes $\vec{r} + d\vec{r}$. Thus,

$$l = 2\mu \frac{dA}{dt}$$

Namely, the angular momentum conservation means a constant areal velocity. This is the general form of **Kepler's second law**, which was discovered for the special form of $U(r) \propto -1/r$.

13.2.2 Energy conservation, and the solution

The Hamiltonian is the next integral of motion. Recall that $H = \sum_i p_i \dot{q}_i - L$.

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

where $p_r = \partial L/\partial \dot{r} = \mu \dot{r}$ has been used as well as the above result for p_θ. This is the energy ($E = T + U$). By using l, we can put it in this form

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)$$

This form nicely fits the general form of the energy that motivates the definition of an effective potential (LN 11, pages 2,3). Going further, one can give a physical interpretation of the effective potential as that potential a particle experiences in the reference frame that rotates with the particle. In such a frame the motion of the particle is governed by the effective potential,

$$U_{eff} = \frac{l^2}{2\mu r^2} + U(r)$$

The first term in the effective potential is the **centrifugal energy**, which prefers the particle to be at large r. The solution for this motion is then, if we do as in LN 3, page 8,

$$t = \pm \int_{r_0}^{r} dr' \frac{1}{\sqrt{\frac{2}{\mu}[E - U(r')] - \frac{l^2}{\mu^2 r'^2}}}$$

Using $d\theta = l dt/(\mu r^2)$, then

$$\theta - \theta_0 = \pm \int_{r_0}^{r} dr' \frac{l}{r'^2 \sqrt{2\mu[E - U(r')] - l^2/r'^2}}$$

Let us analyze the potential energy. Note that the centrifugal energy is a **repulsive energy**, in the sense that the force from it is $l^2/(\mu r^3)$, positive at any r value. Possible motions are determined by the nature of $U(r)$, of course. These are interesting physical cases that we will consider. First, we consider an attractive $U(r)$: $-dU/dr < 0$. Second, $U(r)$ should not be too attractive as $r \to 0$. If $U(r \to 0)$ is too attractive, then all motions will end up at a collision of two bodies at $r = 0$. So, let us restrict to the case when $|U(r \to 0)| <$ the centrifugal term. Third, let us consider the case when $|U(r \to \infty)| >$ the centrifugal term.

These conditions are satisfied by potential energies such as $U = -k/r, kr^2/2$ or kr. The first two are, the Kepler/Newton problem and the Hooke's law problem, respectively, of course. When these conditions are satisfied, there must be a minimum of U_{eff} at some r, since U_{eff} is attractive as $r \to \infty$ (due to $U(r)$), and is repulsive as $r \to 0$ (due to the centrifugal term).

For this class of potential energies, the following lists some typical motions possible. We define turning points as those r values that satisfy $E = U_{eff}$ (cf. Lecture 3).

1. If there is one turning point, and E is equal to the minimum of U_{eff}, then a circular motion (r = constant, $l = \mu r^2 \dot{\theta}$ = constant) occurs.
2. If there are two turning points, r_{min} and r_{max}, and neither of them are equilibrium points for $U_{eff}(r)$, then a bound motion occurs. This is an oscillatory motion in r, accompanied by a non-uniform but unidirectional rotation in θ.
3. If there is only one turning point, but E is greater than the minimum of U_{eff}, then it means an unbound motion. The particle will disappear or "escape" to the infinity eventually.

In case 2, the motion can be thought of as two periodic motions in r and θ. These two periods do not need to be commensurate to each other. When the two periods τ_r and τ_θ are commensurate, though, i.e., when $n\tau_r = m\tau_\theta$ for some non-zero integers n, m, then we have the case of a closed orbit, characterized by a finite period of the overall 2D motion. It can be shown (Bertrand's theorem) that the only power law central forces that can give such a closed orbit motion for $E > U_{eff,min}$ are the $-kr$ force (Hooke's law in 3D) and the $-k/r^2$ force (Newton's law of gravity or Coulomb force for opposite charges).

13.3 Hooke's law problem and the Kepler problem

Let us consider the two most important cases of the central force problem: $U(r) = kr^2/2$ (Hooke's law) and $U(r) = -k/r$ (Kepler problem). As we learned in the last lecture, we have an effectively 2D problem with a conserved angular momentum. The nature of the motion in the Hooke's law case is easy. It is a 2D SHO (Section 3.3 of the book). As shown in Figure 3.2 of the book, the orbits in this case is a circle, an ellipse, or a line, all with the period $\tau = 2\pi\sqrt{m/k}$ just like the associated 1D SHO that we know and love. The math in this case is very similar to what we did for 1D SHOs (pages 104, 105 of the book).

In the remainder of this lecture, we discuss the Kepler problem.

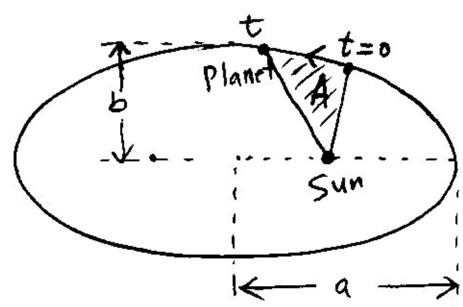
13.4 Kepler's laws

These laws apply to the motion of planets in our solar system. The diagram below is quite **exaggerated** in terms of the difference between a and b. As a matter of fact, most orbits of our planets are close to circles ($a \approx b$).

1. A planet's orbit is an ellipse with the Sun at one of the two focus points. See the diagram, which quite exaggerates the elliptical nature of the orbit. Most orbits are close to circles.

2. The areal velocity dA/dt is constant. See the diagram. The initial point ($t = 0$) can be arbitrarily chosen. A is the cumulative area swept by the position vector of the planet.

3. $\tau^2 \propto a^3$, where τ is the period of the motion, and a is the semi-major axis of the ellipse. a can be replaced by any linear dimension of the ellipse, such as the semi-minor axis b or the mean radius.



13.5 Kepler problem

These observational laws of Kepler can be proven if we use Newton's law of gravity for which

$$U(r) = -k/r$$

where $k = GM_S m_p$, M_s is the mass of the Sun, and m_p is the mass of a planet. Note that the reduced mass, relevant for the relative motion is given by $\mu = m_p M_S/(m_p + M_S) \approx m_p$ in the zero-th order as $M_S \gg m_p$. However, we will keep using μ, below, as this problem can describe any two body problem of celestial bodies. For instance, for a binary star consisting of two rotating equally massive stars, the reduced mass will be half of the individual mass.

The solution for the orbit can be found from the θ equation (the last equation of page 6 with $\theta_0 \equiv 0$ and integrating after a change of variable $u = 1/r$: see the derivation at the end of this LN)

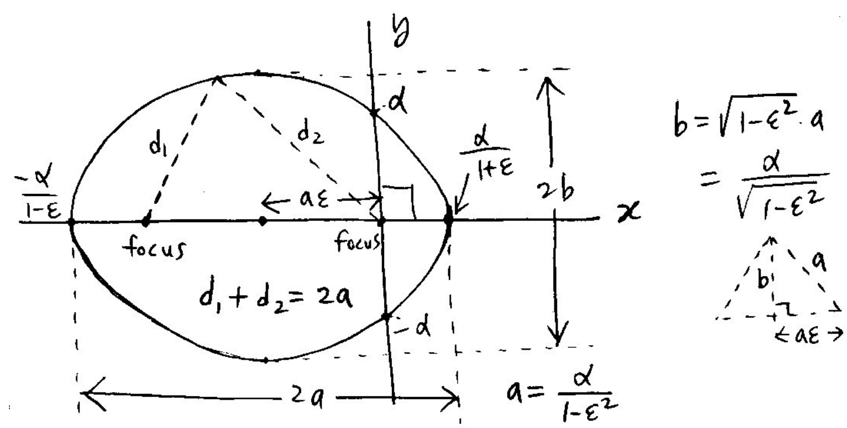
$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta$$

where $\alpha = l^2/\mu k,\ \varepsilon = \sqrt{1 + 2El^2/\mu k^2}$.

The shape of the orbit is the so-called "conic section" and it depends on the **eccentricity** ε. 2α is called the **latus rectum**.

Orbit	Eccentricity (ε)	Energy (E)	Note
Circle	0	$E = U_{eff,min}$	The easiest, the most important!
Ellipse	$0 < \varepsilon < 1$	$U_{eff,min} < E < 0$	
Parabola	1	$E = 0$	Escape condition, open orbit
Hyperbola	$\varepsilon > 1$	$E > 0$	Open orbit

Here is a summary of the geometry for the elliptical orbit and the circular orbit (**a special case when** $\varepsilon = 0$ **and** $a = b$):



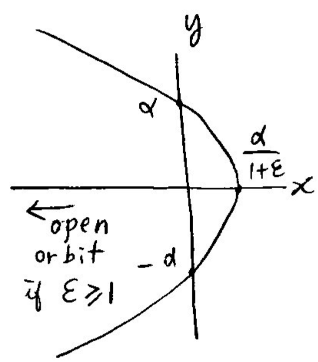
This proves Kepler's first law. The **semi-major axis** a and the **semi-minor axis** b are given by

$$a = \frac{\alpha}{1 - \varepsilon^2} = \frac{k}{2|E|}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$$

In the diagram, the fact that $a = \alpha/(1-\varepsilon^2)$ follows from the fact that $2a = \alpha/(1+\varepsilon) + \alpha/(1-\varepsilon)$. The fact the center of the ellipse lies at $x = -a\varepsilon$ follows from the fact that that x value must be the mean of $\alpha/(1+\varepsilon)$ and $-\alpha/(1-\varepsilon)$ (min/max of r, or apsides, corresponding to $\theta = 0$ and π). Then, $b = \alpha/\sqrt{1-\varepsilon^2}$ follows from the right triangle shown on the right side of the above diagram, considering the case $d_1 = d_2 = a$.

Before we discuss more about the elliptical orbits, here is a rough diagram for the parabolic or elliptical orbit. It can be considered as an elliptical orbit that becomes too large and eventually open up at the $x \to -\infty$ end as $\varepsilon \to 1$ (parabola) and greater (hyperbola).



[For the mathematically curious only...] The actual mathematical "proof" that the above equation $\frac{\alpha}{r} = 1 + \varepsilon\cos\theta$ really leads to all the conic section curves listed above, depending on the value of the eccentricity, is given now. Here, by "proof," I mean rendering the above polar coordinate equation to the "standard" Cartesian coordinate equation. Note that this equation can be written as $\alpha = r + r\varepsilon\cos\theta$ or $\alpha - \varepsilon x = r$. Squaring both sides, we get $(\alpha - \varepsilon x)^2 = x^2 + y^2$. It is immediately obvious that $\varepsilon = 1$ will render this equation to $\alpha^2 - 2\alpha\varepsilon x = y^2$, which represents a parabola. It is also trivial to see that $\varepsilon = 0$ means $r = \alpha$ (circle). It takes simple algebra to prove that $0 \le \varepsilon < 1$ will render the above equation to the form

$$1 = \frac{\left(x + \frac{\alpha\varepsilon}{1-\varepsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2}$$

Here, a and b are as given above. This fits the general equation of an ellipse in the Cartesian coordinate system $(x-x_0)^2/a^2 + (y-y_0)^2/b^2 = 1$. The circle can be considered as a special ellipse, with $\varepsilon = 0$ leading to $a = b$. For $\varepsilon > 1$, we get

$$1 = \frac{\left(x - \frac{\alpha\varepsilon}{\varepsilon^2-1}\right)^2}{\left(\frac{\alpha}{\varepsilon^2-1}\right)^2} - \frac{y^2}{\left(\frac{\alpha^2}{\varepsilon^2-1}\right)}$$

This fits the general equation of a hyperbola in the Cartesian coordinate system $(x - x_0)^2/a^2 - (y - y_0)^2/b^2 = 1$.

Optional Reading "Just Math"

Derivation of

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta}$$

$$\theta = \int_{r_0}^{r} dr' \cdot \frac{\ell}{r'^2\sqrt{2\mu\left(E - U(r')\right) - \frac{\ell^2}{r'^2}}} \Rightarrow$$

$U(r') \to -\frac{k}{r'}$

$$\alpha = \frac{\ell^2}{\mu k}, \quad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

$$\frac{1}{r'} = u, \quad du = -\frac{dr'}{r'^2}$$

$$\theta = -\int_{u_0}^{u} du \cdot \frac{\ell}{\sqrt{2\mu(E + ku) - \ell^2 u^2}}$$

$$\to \sqrt{2\mu E + \frac{\mu^2 k^2}{\ell^2} - \ell^2\left(u - \frac{\mu k}{\ell^2}\right)^2}$$

$$= \ell\sqrt{\frac{2\mu E}{\ell^2} + \frac{\mu^2 k^2}{\ell^4} - \left(u - \frac{\mu k}{\ell^2}\right)^2} \qquad \frac{1}{\alpha} \equiv \frac{\mu k}{\ell^2}$$

$$= \ell\sqrt{(\varepsilon^2)\frac{1}{\alpha^2} - \left(u - \frac{1}{\alpha}\right)^2} \qquad \varepsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu \ell^2}}$$

$$\theta = -\int_{u_0}^{u} du \cdot \frac{1}{\sqrt{\frac{\varepsilon^2}{\alpha^2} - \left(u - \frac{1}{\alpha}\right)^2}}$$

$$u - \frac{1}{\alpha} = \frac{\varepsilon}{\alpha}\cos\xi$$

$$\theta = \cos^{-1}\xi \Rightarrow \xi = \theta$$

$$\frac{1}{r} - \frac{1}{\alpha} = \frac{\varepsilon}{\alpha}\cos\theta \Rightarrow \boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta}$$