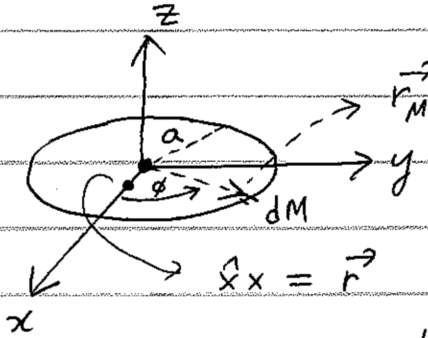
Gravity example (Ex 5,3)

• Consider $\vec{r}$ in the xy plane only.



$$\vec{r}_M = a\cos\phi\,\hat{x} + a\sin\phi\,\hat{y}$$

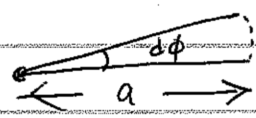
$$\Phi = -\int dM \frac{G}{|\vec{r}-\vec{r}_M|}$$

$$\frac{dM}{a\,d\phi} = \frac{M}{2\pi a} = \text{mass density (linear)}$$

M ← total mass

$2\pi a$ ← circumference

$dM = M\cdot\frac{d\phi}{2\pi}$



$$|\vec{r}-\vec{r}_M| = \sqrt{(a\cos\phi - x)^2 + a^2\sin^2\phi}$$
$$= \sqrt{a^2 - 2a c_\phi x + x^2}$$

$$\Phi = -\frac{MG}{2\pi}\int d\phi \cdot \frac{1}{\sqrt{a^2 - 2ac_\phi x + x^2}}$$

If $|x/a| \ll 1$,

$$\frac{1}{\sqrt{a^2 - 2ac_\phi x + x^2}} = \frac{1}{a}\left(1 - 2c_\phi\frac{x}{a} + \left(\frac{x}{a}\right)^2\right)^{-\frac{1}{2}}$$

$\frac{1}{2}\frac{1}{2}\cdot\frac{3}{2}\cdot 4$

$$\simeq \frac{1}{a}\cdot\left[1 + \frac{1}{4}c_\phi\frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{3}{2}c_\phi^2\left(\frac{x}{a}\right)^2\right]$$

Integrating,

$$\Phi \simeq -\frac{MG}{a}\left(1 + \frac{1}{4}\left(\frac{x}{a}\right)^2\right)$$

One can change $x \to \rho = \sqrt{x^2+y^2}$ considering the symmetry of the prob.

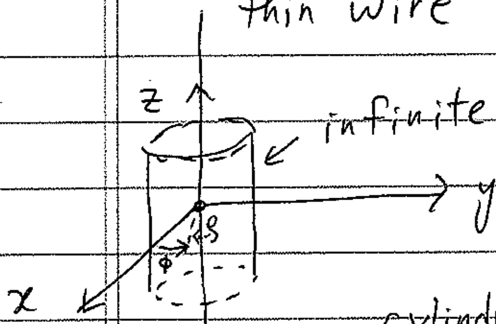
$$\Phi \simeq -\frac{MG}{a}\left(1 + \frac{1}{4}\left(\frac{\rho}{a}\right)^2\right) \Leftarrow \left(\begin{matrix} z=0 \\ \text{case} \\ \text{only}\end{matrix}\right)$$

The origin is an unstable equilibrium point. $\left(\frac{\rho}{a}\right) \ll 1$

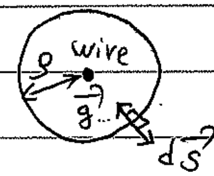
Gravity example #2 Gauss law

thin wire (infinite length)

infinite tube that surrounds the wire



Top view



(ρ, ϕ, z) ← cylindrical coordinates

(ρ = radius of tube)

Considering a tube (infinite length) of radius

$\rho = \sqrt{x^2+y^2}$

and using Gauss law for it

$$\int_{\text{tube}} \vec{\nabla}\cdot\vec{g}\, dv = \int_{\text{tube surface}} \vec{g}\cdot d\vec{s} = -\, g \int dz \int_0^{2\pi} d\phi\, \rho$$

length of tube or wire $L \to \infty$

↑ the magnitude of $\vec{g}$ constant on the tube due to the cylindrical symmetry

$= -g\, L\, 2\pi\rho$

$= -4\pi G M$

M : total mass of the wire $(\to \infty)$

L : length of the wire $(\to \infty)$

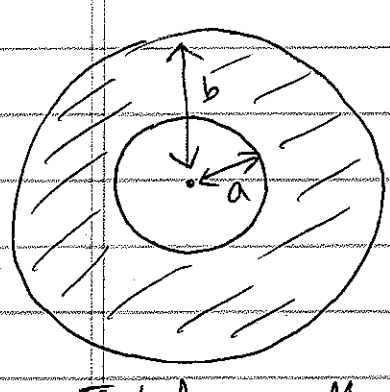
$A \equiv M/L$ = linear mass density of the wire (finite)
= mass per unit length

$g = 2GA/\rho$

$\vec{g} = -\,2GA\,\hat{\rho}/\rho$ — sign means $\vec{g}$ towards the origin

$\Phi = +\,2GA \ln \rho$

Ex 5.1 Spherically symmetric problem.
Gauss law problem!



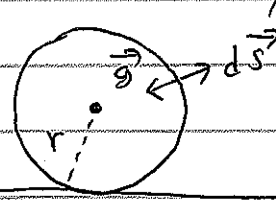
Total mass M

mass between $a < r < b$ only
mass "shell"

Due to the spherical symm.
$g = |\vec{g}|$, Φ depend only on r
not on θ, ϕ

(r, θ, ϕ) --- spherical coord. sys.

Gauss law at any r



$-g \cdot 4\pi r^2 =$ mass inside radius r $\times (-4\pi G)$

① If $r < a$, $g = 0$ $\because$ no mass inside

② If $r > b$, $g = \frac{GM}{r^2}$ $\quad \vec{g} = -\frac{GM}{r^2}\hat{r}$

~~Just as though there is a point mass at the origin!~~

$\Rightarrow$ Generally true for any spherical mass distribution, if the field is measured completely outside the mass distribution.

③ If $a < r < b$, mass inside $r = \frac{r^3 - a^3}{b^3 - a^3} \cdot M$

$$\vec{g} = -\frac{GM}{r^2} \cdot \left(\frac{r^3 - a^3}{b^3 - a^3}\right) \cdot \hat{r}$$

Note that $\vec{g}(r = a) = 0$ --- agrees with ①

$\vec{g}(r = b) = -\frac{GM}{b^2}\hat{r}$ ... agrees with ②

$\therefore$ $\vec{g}$ is continuous everywhere.

why is $\vec{g}$ continuous ?

because mass density is only moderately discontinuous (like a step function)

$\rightarrow$ $\vec{\nabla} \cdot \vec{g} = -4\pi G \rho$ (ρ : mass density)

$\rightarrow$ $\vec{g}$ is an integral of ρ

If ρ is only moderately discontinuous (piece-wise conti like a step ftn)

then $\vec{g}$ will be continuous.

Next, get Φ from $\vec{g} = -\vec{\nabla}\Phi$

① If $r<a$, $\Phi = \text{const.} = C$

② If $r>b$, $\Phi = -\frac{GM}{r} + \text{const} \equiv -\frac{GM}{r}$

Just fixing the over-all offset of Φ. (One way to do it) ←-- by requiring $\Phi(r\to\infty)=0$ ↳ choose to be 0

③ If $a<r<b$, $\Phi = \frac{GM}{b^3-a^3}\left(\frac{r^2}{2} + \frac{a^3}{r}\right) + A$ ← const.

{Note} $\vec{g}$ continuous $\rightarrow$ Φ must be continuous. (actually, differentiable !)

Determine A, and C by continuity.

② ... $\Phi(r\to b) = -\frac{GM}{b}$ ③ ... $\Phi(r\to b) = \frac{GM}{b^3-a^3}\left(\frac{b^2}{2}+\frac{a^3}{b}\right)+A$

Solving for A we get $A = -\frac{3}{2}\cdot\frac{GM}{b^3-a^3}\cdot b^2$

$\therefore$ ③ ... $\Phi = \frac{GM}{b^3-a^3}\left(\frac{r^2}{2} + \frac{a^3}{r} - \frac{3}{2}b^2\right)$

as $r\to a$, this one goes to $\Phi(r\to a) = \frac{3}{2}\cdot\frac{GM}{b^3-a^3}\cdot(a^2-b^2)$

This is the value for ① $r<a$ by continuity.