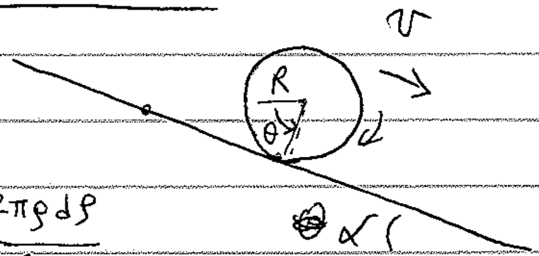
do it

## Ex 7.9




Constraint $v = R\dot{\theta}$

$s = R\theta$

s: distance traveled from rest

$$M\int_0^R \rho^2 \frac{2\pi\rho\, d\rho}{\pi R^2}$$

$U = -Mgs\sin\alpha$

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\cdot\dot{\theta}^2$$

$v = \dot{s}$

$$= \frac{1}{2}M\dot{s}^2 + \frac{1}{4}MR^2\dot{\theta}^2$$

$ds - R\,d\theta = 0$

$$L = \frac{1}{2}M\dot{s}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mgs\sin\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = -\lambda(t)R$$

$\dot{s} = R\dot{\theta}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = \lambda(t)$$

$$\rightarrow \frac{1}{2}MR^2\ddot{\theta} = -\lambda(t)R$$

$$\rightarrow M\ddot{s} - Mg\sin\alpha = \lambda(t) = -\frac{1}{2}MR\ddot{\theta}$$

$MR\ddot{\theta}$

$$\boxed{\ddot{\theta} = \frac{2\sin\alpha}{3R}g}$$

$\rightarrow 0$ as $\alpha \rightarrow 0$

Non-sense!!

This model is highly idealized. Not enough for real rolling wheels.

$$\lambda(t) = -\frac{1}{3}Mg\sin\alpha \leftarrow \text{friction}$$

$$-\lambda(t)R = \frac{1}{3}MgR\sin\alpha \leftarrow \text{torque}$$

# Notes for Lecture 11

# Effective potential, gravity

## 11.1 Effective potential

Let us consider a problem for which the Lagrangian is not explicitly dependent on time. So, the Hamiltonian is conserved. Suppose that the Hamiltonian can be written as $\frac{1}{2}m\dot{x}^2 + f(x)$ where $f(x)$ is a function of $x$ and other parameters of the problem, including any other conserved quantities (momentum, angular momentum, etc). In this case, we can consider $f(x)$ as our "effective potential" and write

$$H = \frac{1}{2}m\dot{x}^2 + U_{eff}(x)$$

- Just because the Hamiltonian can be written in terms of only one generalized coordinate does not mean that the Lagrangian contains only one generalized coordinate. The actual motion may very well involve multiple degrees of freedom, and thus multiple generalized coordinates. Still, the Hamiltonian can end up depending only on one generalized coordinate and its time derivative if there are other constants of motion (conserved quantities).

- Note that under these conditions, the EOM is easily integrable (LN5 page 9). Also, some key elements of the motion such as turning points and the period can be easily written down (LN 5).

- Here, $x$, $\dot{x}$ are stand-ins for a generalized coordinate $q$, $\dot{q}$. It can be other variable such as $\theta$, depending on the nature of the problem. Likewise, $m$ can be other quantities such as the rotational inertia.

- Note that $m$ is, in any case, an *effective* mass/rotational-inertia in general (cf. Homework 5.4). Moreover, $m$ can be even dependent on $x$: $m = m(x)$ (cf. Homework 6)! This does not pose any additional problem for formally integrating the equation,

$$H = \frac{1}{2}m(x)\dot{x}^2 + U_{eff}(x) = \text{const}$$

$$t = \pm \int_{x_0}^{x} dx' \sqrt{\frac{m(x')}{2(H - U_{eff}(x'))}}$$

  or any other general properties (turning points, equilibrium points, etc), as long as $m(x)$ is well-behaved (positive definite, for example).

- Thus, one can write

$$H = T_{eff} + U_{eff}$$

  where $T_{eff} \equiv \frac{1}{2}m(x)\dot{x}^2$ is the effective kinetic energy and $U_{eff}$ is the effective potential energy. This does *not* mean, in general, that $L = T_{eff} - U_{eff}$. It may be in many cases, but it may not be in many other cases. Since $L$ is hardly a conserved quantity, whether $L$ can be written in this fashion or not is less of a practical concern.

## 11.2 Gravity

### 11.2.1 Newton's law of gravity

For two bodies, there is an attractive force of the magnitude

$$F = G\frac{Mm}{r^2}$$

and the direction which is parallel to the line joining the two bodies. Here, $M$, $m$ are the masses of the two bodies. $r$ is the distance between them.

### 11.2.2 Gravitational field

Field is an important modern concept. It does away the "action at a distance," which Newton himself had a hard time believing (and so did Einstein).

Consider a body of mass at $M$ found at some point. Let us conveniently take that point to be the origin. Then, we say that the gravitational field, $\vec{g}$, at position vector $\vec{r}$ due to this mass $M$ is

$$\vec{g} \;=\; -\frac{GM}{r^2}\hat{r}$$

where $r = |\vec{r}|$, and $\hat{r}$ is the radial unit vector $\hat{r} = \vec{r}/r$. Observe that taking the position of mass $M$ was purely for convenience. In general, we can change $\vec{r} \to \vec{r} - \vec{r}_M$, and $\hat{r} \to (\vec{r} - \vec{r}_M)/|\vec{r} - \vec{r}_M|$, where $\vec{r}_M$ is the position of mass $M$, and everything is good.

Then, the force that another mass $m$ feels due to $M$ is given by

$$\vec{F}_m = m\vec{g}$$

Two comments. (1) We are defining $\vec{g}$ generally here, not just the Earth gravity near its surface. In general, $\vec{g}$ is position dependent, not constant. (2) The definition of a field is a mathematical triviality, at this level. But, imagine that the field is some sort of a real thing that connects two massive bodies! The concept of the field is a big deal, while the particles we think are responsible for the gravitational field (gravitons) haven't been detected by any human scientific equipment yet (compare this situation with photons which are responsible for the electromagnetic field).

### 11.2.3 Gravitational potential

Let us look at the mathematics of the field a bit: $\vec{g}(\vec{r}) = -GM\hat{r}/r^2$. This is a conservative field. Which means two things:

$$\vec{\nabla} \times \vec{g} = 0$$

$$\vec{g} = -\vec{\nabla}\Phi(\vec{r})$$

Here, $\Phi$ is related to the potential energy $U$ via $U = m\Phi$, where $m$ is the other mass that interacts with mass $M$. The gravitational potential $\Phi$ can be obtained as

$$\Phi(\vec{r}) \;=\; -\frac{GM}{r}$$

Indeed, the existence of such a potential proves that the gravity is a conservative force. Or, generally, if the mass $M$ is not at the origin, but at $\vec{r}_M$:

$$\Phi(\vec{r}) \;=\; -\frac{GM}{|\vec{r} - \vec{r}_M|}$$

Let us consider a simple fact. If there are multiple bodies, then the total force is obviously the addition of all forces. Each force can be considered as coming from a potential field. It then follows that the potential field is also additive. This is due to the linear operator nature of the $\vec{\nabla}$ operator that connects the potential and the field. So, for a gravitational field that arises from multiple bodies,

$$\begin{aligned}\Phi(\vec{r}) &= -G\sum_i \frac{M_i}{|\vec{r}-\vec{r}_{M_i}|}\\ \vec{g} &= -G\sum_i \frac{M_i}{|\vec{r}-\vec{r}_{M_i}|^3}(\vec{r}-\vec{r}_{M_i})\end{aligned}$$

For a continuous distribution of masses, these change to the integral:

$$\begin{aligned}\Phi(\vec{r}) &= -G\int \frac{dM}{|\vec{r}-\vec{r}_M|}\\ \vec{g} &= -G\int dM\frac{\vec{r}-\vec{r}_M}{|\vec{r}-\vec{r}_M|^3}\end{aligned}$$

### 11.2.4 Gauss law, Poisson equation

Let us consider a volume integral

$$\int_V dV\vec{\nabla}\cdot\vec{g} = \int_S d\vec{S}\cdot\vec{g}$$

The above equality of the volume integral of a divergence of a vector field and the surface integral of the vector field is called **Gauss's theorem**. Very important. Here, $S$ is the surface area of the volume $V$, and $d\vec{S}$ is a small area element vector. The magnitude of $d\vec{S}$ is the area of the area element, and the direction of it is normal to the area element and towards the outside of the volume.

Consider the simplest case first. Assume $\vec{g} = -GM\hat{r}/r^2$, and the volume $V$ is a sphere of a radius $R$, centered at the origin. The integral is then

$$\begin{aligned}\int_V dV\vec{\nabla}\cdot\vec{g} &= \int_S d\vec{S}\cdot\vec{g}\\ &= -\frac{GM}{R^2}\int_0^{\pi} d\theta\int_0^{2\pi} d\phi\sin\theta\ R^2\\ &= -GM\int d\Omega\\ &= -4\pi GM\end{aligned}$$

Here, $d\Omega$ is the infinitesimal solid angle[1]

$$d\Omega \stackrel{def}{=} \frac{d\vec{S}\cdot\hat{r}}{r^2}$$

subtended by the area element $d\vec{S}$ at the origin. Using the concept of the solid angle, the above integral immediately generalizes to any volume, which may or may not enclose the origin:

$$\begin{aligned}\int_V dV\vec{\nabla}\cdot\vec{g} &= \int_S d\vec{S}\cdot\vec{g} \\ &= -GM\int d\vec{S}\cdot\hat{r}/r^2 \\ &= -GM\int d\Omega\end{aligned}$$

Note that whenever the volume $V$ encloses the origin, then it is $\int d\Omega = 4\pi$, while if $V$ does not enclose the origin, then $\int d\Omega = 0$.

$$\begin{aligned}\int_V dV\vec{\nabla}\cdot\vec{g} &= -4\pi GM \qquad \text{if } V \text{ encloses the mass } M\text{, the source of } \vec{g}, \\ &= 0 \qquad \text{if it does not,}\end{aligned}$$

for *any volume* $V$.
What if the mass $M$ is not placed at the origin? In that case, $\int dV\vec{\nabla}\cdot\ldots = \int dV_M\vec{\nabla}_M\cdot\ldots$ by a mere translation of the coordinate vectors, where the subscript $M$ means the coordinate system whose origin is at $\vec{r}_M$. Therefore, the above result is valid even if $M$ is displaced from the origin. [**Note 1**: This simply means that if we shift the position of the mass and the volume of integration at the same time, the integral should not change at all. This is obvious since we are free to choose the origin at any point that we like. **Note 2**: The key point to remember here is that no matter how we deform the volume it will remain invariant as long as the mass stays inside the volume, if the initial volume enclosed the mass, or the mass stays outside the volume, if the initial volume did not enclose the mass. For instance, suppose we start from a sphere with a mass $M$ at the origin of the sphere. If we shift the sphere a little so that now the mass $M$ is off-center with respect to the sphere, the above integral remains invariant as long as $M$ continues to stay inside the sphere.]

[1] This is the general definition of the infinitesimal solid angle for an area element $d\vec{S}$ which is at the position vector $\vec{r}$. Note that it can be positive or negative depending upon whether the normal vector of the area element is pointing away from the origin or towards it. This should not be surprising, given the fact that the "linear" angle also has a sign. In the spherical coordinate, the volume element $dV = r^2\sin\theta\, drd\theta d\phi = r^2 drd(\cos\theta)d\phi$. This is **always** equal to $dV = r^2 drd\Omega$. This may be used as an alternative definition of $d\Omega$, if you like, and is equivalent to the above definition.

Therefore, the above result then immediately generalizes to the case when there is any distribution of masses, not just one mass.

So, what we have is the generalization of the above result:

$$\int_V dV \vec{\nabla} \cdot \vec{g} \;=\; \int_S d\vec{S} \cdot \vec{g} \;=\; -4\pi G M$$

$V$ is *any volume.*
$M$ is the total mass enclosed by $V$, including 0.
$\vec{g}$ is the total gravitational field, due to *all masses* around, not just $M$.

Since $V$ is any volume, it can be taken to be $dV$. Then, $M = \rho(\vec{r})dV$, where $\rho(\vec{r})$ is the mass density. Then, we have

$$\vec{\nabla} \cdot \vec{g} \;=\; -4\pi G \rho(\vec{r})$$

These two boxes above contain an extremely useful law of physics: the **Gauss's law** for the gravity. Gauss's law is applicable to Newton's law of gravity and Coulomb's law of the electrostatic force.

Gauss's law can be re-written in terms of $\Phi$,

$$\vec{\nabla}^2 \Phi \;=\; 4\pi G \rho(\vec{r})$$

This is the **Poisson's equation** for the potential $\Phi$. It is a kind of differential equation, like the SHO ODE that we solved.

For a point mass at $\vec{r}_M$, $\rho(\vec{r}) = M\delta(\vec{r} - \vec{r}_M)$, where $\delta(\vec{r} - \vec{r}_M)$ is the Dirac-delta function in 3 dimensions, note that we know the solution $\Phi$ already. It is $-GM/|\vec{r} - \vec{r}_M|$. So, we can write

$$-\vec{\nabla}^2 \frac{1}{4\pi|\vec{r} - \vec{r}_M|} \;=\; \delta(\vec{r} - \vec{r}_M)$$

That is, $-\frac{1}{4\pi|\vec{r}-\vec{r}_M|}$ is the **Green's function** of the Poisson equation.

### 11.2.5 Meaning of the gradient

Suppose you make a plot of equipotential lines, i.e. a collection of curves, which satisfy $\Phi$ = constant. Where does the force field, $\vec{g}$ point? Due to the nature of the gradient, $\vec{g}$ always points perpendicular to the equipotential line. This does not uniquely determine the direction of $\vec{g}$. It could point along the direction in which the potential increases, or the direction in which the potential decreases. As $\vec{g}$ is the negative gradient of $\Phi$, $\vec{g}$ points towards the direction in which $\Phi$ decreases. Look at Figure 5.8 of the textbook, and figure out which way the force field is pointing at some points. Also, note that the force field is greater in magnitude where the equipotential lines are dense.