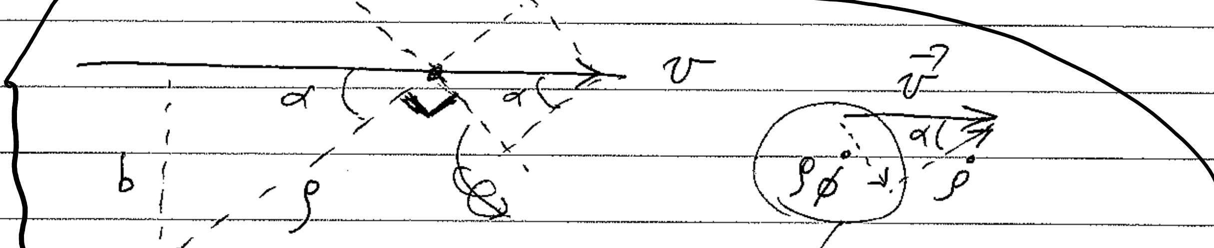
related to the last LN. L10 - 1



$$\rho\dot{\phi} = v\sin\alpha = v\cdot\frac{b}{\rho}$$

$$L_\phi = m\rho^2\dot{\phi} = mvb$$

direction $\rightarrow \otimes$

§. Hamiltonian Mechanics

$$dL = \underbrace{\overset{\dot{p}}{\frac{\partial L}{\partial q}}dq + \overset{p}{\frac{\partial L}{\partial \dot{q}}}d\dot{q}} + \frac{\partial L}{\partial t}\overset{dt}{\cancel{\ }}$$

Consider $H = p\dot{q} - L$ (Legendre transformation)

$$dH = dp\,\dot{q} + p\,d\dot{q} - dL$$

$$= \dot{q}\,dp - \dot{p}\,dq - \frac{\partial L}{\partial t}dt$$

[indep. variables $q, \dot{q}, t \Rightarrow q, p, t$]

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\boxed{\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\dot{p}}$$

Canonical equations of motion

$$\boxed{\frac{dH}{dt} = \dot{q}\dot{p} - \dot{p}\dot{q} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}}$$

For many degrees of freedom

$$\boxed{H = \sum_i p_i\dot{q}_i - L} \qquad \boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i}$$

§. Principle of (energy or H) conservation.

$$\frac{dH}{dt} = 0 \quad \text{if } L \text{ (or } H\text{) does not depend on } t \text{ explicitly.}$$

(homogenity of time)

H : hamiltonian

proof
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$
prev. page

It is the "energy" in most, but not all, cases.

If $L = \frac{1}{2} m \vec{v}^2 - U(\vec{r})$

$$\frac{\partial L}{\partial \dot{\vec{r}}} = m\vec{v} = \vec{p}$$

$$\begin{aligned} H &= \vec{p} \cdot \vec{v} - \frac{1}{2} m \vec{v}^2 + U(\vec{r}) \\ &= \vec{p} \cdot \frac{\vec{p}}{m} - \frac{\vec{p}^2}{2m} + U(\vec{r}) \\ &= \frac{\vec{p}^2}{2m} + U(\vec{r}) \\ &= T + V \end{aligned}$$

Note : H is a function of p, q not of $\dot{q}, q$

$$L(q, \dot{q}, t) \longrightarrow H(p, q, t)$$

$$\longrightarrow p\dot{q} - L \longrightarrow$$

§. Some words about H and other aspects of the Lagrangian Mechanics

① The canonical EoM $\dot{p} = -\frac{\partial H}{\partial q}$, $\dot{q} = \frac{\partial H}{\partial p}$ are not used often to do problems. (Lagrangian does just fine.) Rather, the Hamiltonian dynamics is important as a formalism that connects to one of the common formalism of QM --- the (Heisenberg) formalism. One can say that if one quatizes the Poisson bracket (see homework) then the QM results.
(Schrödinger)

② H is important in the sense that it describes mechanics in the "true phase space," (p, q). p and q are canonically "conjugate variables".

For instance, the Louiville's theorem holds in the (true) phase space. (Read 7.12) The essence of this theorem is that if you prepare an "ensemble" of many systems which have different initial conditions then the density of the points that represent those systems in the phase space is constant. (Important in charged particle optics, cosmology, etc..)

③ Virial theorem (7.13)

For power-law force, pot. $U \propto r^n$

$$2\langle T\rangle = n\langle U\rangle$$

average over a long time or a period

§. Lagrangian EOM with constraint(s)

Let us consider one constraint only.
(generalization to multiple constraints straight-f.)

$\star$ Assume $\underbrace{\sum_k a_k \dot{q}_k + a_t = 0}_{\text{a constraint that satisfies}}$.

a_k, a_t → functions of q_k, t

In general, this cannot be integrated ("non-holonomic")
If it can be integrated, then we have a holonomic constraint.
(e.g. $v = R\omega$ for a rolling w/o slipping)

The above equation can be written as

$$\sum_k a_k \, dq_k + a_t \, dt = 0$$

For a virtual displacement $\delta t = 0$

$$\sum_k a_k \, \delta q_k = 0 \quad \cdots\cdots (\star)$$

$$\int_1^2 dt \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k = 0$$

Add $\underset{(\star)}{0}$ times $\lambda(t)$ --- Lagrange undetermined multiplier

$$\int_1^2 dt \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} + \lambda(t) a_k \right] \delta q_k = 0$$

Not all q_k's are independent!

A special commonplace sub-class of all possible constraints $F(q_k, \dot{q}_k, t) = 0$ (all k's $1 \cdots n$)

Holonomic constraint: $F(q_k, \dot{q}_k, t) = G(q_k, t) = 0$

simple. Can reduce the # of the generalized coordinates by 1 per constraint.

But only one constraint.

From $k = 1, \ldots, n$, if we ~~remove~~ exclude (n = D.O.F.)

one of q_k's, then the rest are independent!

So, consider $q_2, \ldots, q_n$ as indep. vars.

As for q_1, <u>choose</u> $\lambda(t)$ so that

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} + \lambda(t)\, a_1 = 0.$$

As for other q_k's, ($k=2, \ldots, n$) the variational principle as before

$$\Rightarrow \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \lambda(t)\, a_k = 0$$

Result? This equation is valid for <u>all</u> k's including $k=1$.

★

Price? $n+1$ unknowns $q_k, \lambda(t)$ $_{k=1,\ldots,n}$

Only n Lagrange eq's ?! No worries. We have the constraint equation

$$\sum_k a_k \dot{q}_k + a_t = 0$$

Why write in this form??

See L8-⑧!

Meaning of $\lambda(t)$: <u>generalized force of constraint</u>

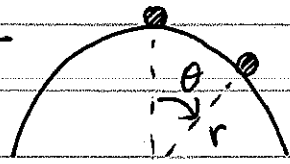
$$-\frac{\partial L}{\partial q_1} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \boxed{+\lambda(t)\, a_1}$$

$-\frac{\partial L}{\partial q_j} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = Q_j$ generalized force

★

Lagrangian mechanics is cool/sneaky in this regard. No need to know the constraint force to start with. It is an <u>outcome</u>! (friction ... normal force ...)

Ex 7.10



$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$U = mgr\cos\theta$$

Why is $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$?

(use $x = r\cos\theta$, $y = r\sin\theta$
$\Rightarrow$ show $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$)

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

Constraint : $r = R \Rightarrow dr = 0$

$$a_r dr + a_\theta d\theta + a_t dt = 0$$
$$a_r = 1, \quad a_\theta = a_t = 0$$

EoM's

① $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda(t) a_r = \lambda(t)$

$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda$

② $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda(t) a_\theta = 0$

$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) - mgr\sin\theta = 0$

$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mgr\sin\theta = 0$

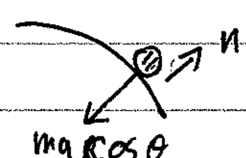
③ $r = R$ (constraint itself)

Now, use the constraint to simplify EoM's

① $\rightarrow \lambda = -mR\dot{\theta}^2 + mg\cos\theta$

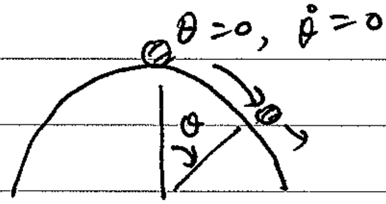
② $\rightarrow \ddot{\theta} = \frac{g}{R} s_\theta$ ⇐ The EoM that we would have gotten had we used the L formalism w/o the constraint.

① $\rightarrow$ $\lambda = ?$



$mg\cos\theta - n = \underbrace{mR\dot{\theta}^2}_{\text{centripetal force}}$ $\rightarrow$ $\therefore \lambda = n =$ normal force

Let us ask the question.



when does the thing take off ?

Ans. when $\lambda = 0$ (no contact force)

$$\Rightarrow \quad mg\cos\theta = mR\dot\theta^2 \quad \text{--- ①}$$

Note that this problem conserveds the energy. How do we know it ?

Note that this L is L from the original formalism. Not the L + constraint formalism. See point ③ of the next page !!

$$\Rightarrow L = \frac{1}{2} m R^2\dot\theta^2 - mgR\cos\theta \quad \Leftarrow \text{no } t\,!$$

$$H = E = \frac{1}{2} m R^2\dot\theta^2 + mgR\cos\theta$$

$$= mgR \quad \left(\begin{array}{l}\text{when } \theta = 0^\circ \\ \text{we assumed } \dot\theta = 0\end{array}\right)$$

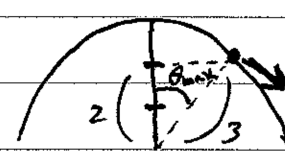
②

Combining ① & ② to eliminate $\dot\theta^2$

$$\Rightarrow \quad \frac{1}{2} mgR\cos\theta + mgR\cos\theta = mgR$$

$$\cos\theta = \frac{2}{3}$$

$$\theta_{max} = \cos^{-1}\frac{2}{3}$$



- Was this easier than doing $\left(\begin{array}{l} mg\cos\theta = m\dfrac{v^2}{R} \\ \dfrac{1}{2}mv^2 + mgR\cos\theta = mgR\end{array}\right)$?

No !

§. L + constraint (one constraint)

$$\sum_k a_k dq_k + a_t dt = 0$$

EoM $$-\frac{\partial L}{\partial q_k} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = a_k \lambda(t)$$

§. L + constraints $i = 1, \cdots, m$ constraints.

$$\sum_k a_{k,i} dq_k + a_{t,i} dt = 0$$

EoM $$-\frac{\partial L}{\partial q_k} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_i a_{k,i} \lambda_i(t)$$

To keep in mind.

① Treat all q_k's as though they are dynamical variables before setting up EoM's.
(t-dep) ← Absolute must!!

② After setting up EoM's, one can use constraints to simplify EoM's!
(Using constraints before setting up EoM's is breaking a rule, big time!! → Not allowed.)

③ Conservation principles ~~rules~~ do not apply to L if using L + constraint(s) formalism.
(Why not? Because conservation principles were derived from $-\frac{\partial L}{\partial q_k} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0$!)