Green's function method for SHO

This is an optional topic – the method of Green's function for solving the SHO problem – highly recommended for your reading, if you have time. Specifically, this shows how one can obtain a particular solution for any given driving force. The following is formatted like a homework problem. The contents of the problem contain all major results within this method. The "solutions" are downloadable on the same web page, where this note is available.

GF method Consider a SHO (simple harmonic oscillator) driven by an external force. Assume that $f(t) = F(t)/m$ is an arbitrary function.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x \;=\; f(t)$$

Here, $\beta \geq 0$, able to represent any physical cases (free, underdamped, critically damped and overdamped SHO). (As usual) we will be concerned with the particular solution only. In this problem, we will be looking for a particular solution that satisfies the following boundary condition

$$x(-\infty) = \dot{x}(-\infty) = 0$$

That is, the system is "quiet" at the far end of the past, before any $f(t)$ is "turned on."

(a) Let us write $x(t) = \exp(\alpha_1 t)g(t)$ where α_1 is one of the solutions for the characteristic equation for the above ODE (ordinary differential equation):

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0.$$

Find the ODE satisfied by $g(t)$. It should be directly integrable over time. Do the integration from $-\infty$ to t, and use the above boundary condition, to show that $g(t)$ satisfies the following ODE, where α_2 is the other solution (which could be identical with α_1) of the above characteristic equation.

$$\dot{g} + (\alpha_1 - \alpha_2)g \;=\; \int_{-\infty}^{t} ds f(s) e^{-\alpha_1 s}$$

(b) Now, define $h(t)$ by

$$g(t) = h(t)e^{(\alpha_2 - \alpha_1)t}$$

and show that the ODE for $h(t)$ is directly integrable. Show that $h(t)$ is given by

$$h(t) = \int_{-\infty}^{t} ds \int_{s}^{t} dt' e^{(\alpha_1 - \alpha_2)t'} e^{-\alpha_1 s} f(s)$$

[Hint: You need to use $\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} ds \cdots = \int_{-\infty}^{t} ds \int_{s}^{t} dt' \cdots$. Draw a diagram to show that this is true.]

(c) Show that $x(t)$ can be written as

$$\boxed{x(t) \;=\; \int_{-\infty}^{t} dt'\; G(t-t') f(t')}$$

where

$$\boxed{G(t-t') \;=\; \theta(t-t') \int_{0}^{t-t'} du\; e^{(\alpha_1-\alpha_2)u}\; e^{\alpha_2(t-t')}}$$

and $\theta(t-t')$ is the Heaviside step function (1 if $t > t'$ and 0 if $t < t'$).

(d) Show that if $f(t) = \delta(t-t_0)$ (the Dirac delta function – please look it up if you are not familiar with it), then

$$x(t) \;=\; G(t-t_0)$$

Such a function – the response of a system upon a unit delta function impulse – is generally called the **Green's function**. It is a concept that permeates throughout all physics. [Note: $f(t)$ should have the physical dimension of force/mass, different from 1/time. So, what does it mean that $f(t) = \delta(t-t_0)$? We assume that we have chosen the units so that the strength (i.e. the integral) of $f(t)$ is 1.]

(e) Show that for an under-damped or free SHO

$$G(t-t') = \frac{\theta(t-t')}{\omega_1} e^{-\beta(t-t')} \sin(\omega_1(t-t'))$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

(f) Show that for a critically damped SHO

$$G(t-t') = \theta(t-t')(t-t') \exp(-\beta(t-t'))$$

(g) Show that for an over-damped SHO

$$G(t-t') = \frac{\theta(t-t')}{\gamma} e^{-\beta(t-t')} \sinh(\gamma(t-t'))$$

where $\gamma = \sqrt{\beta^2 - \omega_0^2}$.

This problem presented an important general method to obtain a particular solution for the driven SHO problem. By plugging in $f(t) = f_0 e^{i\omega t}$, $x(t)$ for a sinusoidal force can be re-derived. Any other function $f(t)$ can be readily handled.

Notes for Lecture 8

Principle of least action

8.1 Principle of least action

Maybe a catchy name: the principle of least action (PoLA). It is also called **Hamilton's principle**.

In this course, this principle is completely equivalent to Newton's laws. From a more general perspective, this principle seems "deeper" or "more elegant" than Newton's law, because it can be applied to other branches of physics as well, like optics, electro-magnetics, and the relativity, when Newton's laws as we learned them becomes impossible, or cumbersome, to generalize. For instance, the well-known **Fermat's principle**, that the light travels in a path that minimizes the time, can be derived from the principle of least action in optics (Homework problem). Also, it is much clearer to see "how quantum mechanics arises" in this view (especially in Feynman's view of quantum mechanics), while, in general, the same task is viewed as impossible if one starts from Newton's laws point of view.

The PoLA means that the following integral, the so-called **action**,

$$S[q(t)] \;=\; \int_1^2 dt\; L(q(t), \dot{q}(t), t)$$

is stationary when $q(t)$ is the actual motion. Let me analyze this sentence one by one. But, first of all, note that the integration range is written as $\int_1^2$, to mean going from t_1 to t_2, not from value 1 to value 2.

1. $S[q(t)]$ defines a **functional**. The square bracket [ ] is used to emphasize the functional nature of S. It means that S takes a function $q(t)$ as input and

returns a number as output. In contrast, a function takes a number and returns a number. **For Hamilton's principle, only those $q(t)$'s for which $q(t_1) \equiv q(1)$ and $q(t_2) \equiv q(2)$ are fixed are considered.** $q(1)$ and $q(2)$ are initial and final positions, in terms of the "generalized coordinate" q. In the integral, $q(t)$ can be any trajectory, physical or hypothetical. Having clearly understood that S is a functional, please do not be surprised, should you see $S(q)$ instead of $S[q]$, when the context is clear that S is a functional.

2. $q(t)$ is a **generalized coordinate**. Here, we only consider one degree of freedom, and so there is only one q. Soon, we will consider many degrees of freedom, in which case we will consider q_i's. The most common example of q is x (or y or z). Or, the angle θ, as in the simple pendulum problem. In general, q can be any function of the linear or the angular coordinates, time and any other parameters of the problem, as long as $L = L(q, \dot{q}, t)$, i.e. the system is fully specified by the generalized coordinate, its time derivative and the time. **So, there will be as many generalized coordinates as the degrees of freedom.** As such, q does not need to have the dimension of length. It can be dimensionless (e.g. angle), or it may have other dimensions (the dimension of the momentum, the angular momentum, the energy, etc). A clever choice of q can make a certain property of the problem obvious (cf. a Homework problem).

3. L is the so-called **Lagrangian**. It is defined as $T - U$, where T is the kinetic energy and U is the potential energy.

4. That $S[q(t)]$ is stationary means the following. Suppose that the true motion occurs along $q_T(t)$. Now, imagine adding a small **variation** $\delta q(t)$ to $q_T(t)$. Consider the subsequent change of S, $\delta S[q_T(t)] \overset{def}{=} S[q_T(t) + \delta q_T(t)] - S[q_T(t)]$. That $S[q(t)]$ is stationary at $q_T(t)$ means that

$$\delta S[q_T(t)] = 0$$

   As we are just beginning this topic, here we distinguished between the general $q(t)$ and the true motion $q_T(t)$. From now on, however, we will not be making such distinctions, assuming that the context makes it clear when $q(t)$ is an arbitrary[1] one, and when $q(t)$ is the true motion.

Exercise (1) Consider the motion of the free particle, $x = x_0 + v_0 t$. Consider $q(1) = x(1) = 0$ and $q(2) = x(2) = v_0$ (and so we have chosen $t_1 = 0$ and $t_2 = 1$). Show that $\int_1^2 L dt$ is indeed minimum for the true path, by examining the integral for $x = x_0 + v_0 t + f(t)$, where $f(t)$ is any function that satisfies $f(1) = f(2) = 0$. (2) Consider a simply harmonic oscillator with $\omega_0 = 1$. Choose $t_1 = 0$, $t_2 = 2\pi$,

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[1] Arbitrary, as long as it is reasonably "nice," e.g. it has a continuous first derivative

$q(1) = x(1) = 0$, $q(2) = x(2) = 0$. In this case, the physical solution can be taken as $\sin t$. Choose *one* differentiable function $f(t)$ (which is not proportional to $\sin t$) that satisfies $f(1) = f(2) = 0$ and show that $S[\sin t + f(t)]$ is ~~greater than~~ stationary with respect to $S[\sin t]$, the action integral for a physical motion. ~~[Proving this for any function $f(t)$ is possible, also, but it is a much more involved process.]~~

8.2 Characteristics of PoLA

PoLA = Hamilton principle. The resulting EOM for L is called the Lagrangian mechanics, as opposed to the Newtonian mechanics.

1. It is a misnomer. Action is not always minimized (for a general potential; the above examples in Exercise are for simple non-general potentials). Rather, $\delta S = 0$ is all we mean and all we need. Minimum, maximum, or saddle point of the action functional – all of these are OK, and do occur[2]. Maxima do not occur in classical mechanics, but it can in other fields (relativistic mechanics).

2. It deals with scalar quantities (action, Lagrangian) rather than vector quantities. It makes life easier.

3. It is equivalent to, but more general than, the Newtonian formulation of classical mechanics. It applies almost everywhere in physics. When it does not, it is generalized easily (the Feynman path integral formalism of quantum mechanics).

4. It is mathematical. If a free body diagram (Newtonian mechanics) provides forces in full view, the Lagrangian mechanics appears to be just a bunch of math equations, which may seem to "hide all physics under the rug." Newtonian mechanics forces you to think. Lagrangian mechanics may seem to force you to just do the math. Whether this is good or bad is up to you. One should always reflect upon the physics of the solution, though!

5. When bodies are in contact and the motions are constrained, the "force of constraint" (normal force, friction, ...) comes out of the mathematical formalism, "even if you don't try very hard."

6. The basic formalism applies only when there exists a potential function U, but the formalism can be extended when there are dissipative forces, using the concept of Raleigh dissipation function (cf. Landau or any other higher level

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[2] However, if t_1 and t_2 are different infinitesimally, then the minimum holds.

mechanics text). We will not deal with such forces within this new formalism in this course, except when they play the role of constraint forces. The fact that such a dissipative process must be put in by hand in addition to the above Hamilton's principle should not be viewed as weakening the case of this new formalism. As dissipative processes are not purely mechanical (they involve heat and thus require statistical physics), this new formalism can be viewed as separating mechanical matters and other matters. Another fact is that even in Newtonian mechanics, dissipative forces are put in by hand using empirical laws.

2 min. "Vista break"

§. QM (and optics) in a nut-shell

<u>all</u> paths are possible!!

$S=\int_1^2 dt\, L$

$\hbar$ (planck const. $h/2\pi$)

Consider all possible paths and calculate $e^{iS/\hbar}$ for each random path.

Sum up all of them (Feynman path integral)

⇒ Get probability amplitude of a particle going from $\underset{(t_1, q_1)}{1} \rightarrow \underset{(t_2, q_2)}{2}$.

For a <u>classical</u> particle the path for $\delta S = 0$

gives a singular contribution

so that is the only motion possible.

<u>NOT</u> in quantum or optics.

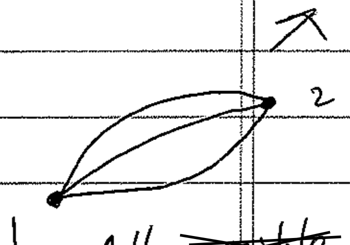
— The End of break —

§. Lagrangian EOM

$$S = \left[\int_1^2 dt\,(T-U) = \right]\int_1^2 dt\,L$$

$$\delta S = \int_1^2 dt\left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta\dot q + \underset{\substack{\parallel\\0}}{\boxed{\frac{\partial L}{\partial t}\delta t}}\right] = 0$$

by definition of a virtual displacement δq.



All ~~possible~~ paths

$\delta q(1) = \delta q(2) = 0$

End points are fixed.

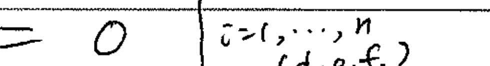
$$= \int_1^2 dt\left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\frac{d}{dt}(\delta q)\right]$$

$$= \int_1^2 dt\,\frac{\partial L}{\partial q}\delta q + \underbrace{\frac{\partial L}{\partial \dot q}\delta q\Big|_1^2}_{\substack{\parallel\\0}} \cancel{q} - \int_1^2 dt\,\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\delta q$$

by definition $\delta q(1) = \delta q(2) = 0$

$$= \int_1^2 dt\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right]\delta q = 0$$

δq is arbitrary $\Longrightarrow$



non-zero only ~~at~~ near a certain value of t
any

Lagrange (Euler) Equation! ←

$$\therefore\quad \boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = 0}$$

at any time !

Generalized to many degrees of freedom →

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = 0} \quad i = 1, \dots, n \text{ (d.o.f.)}$$

- What if there are forces not expressible as a Lagrangian ??

"generalized force"

$$\boxed{-\frac{\partial L}{\partial q_j} + \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} = Q_j}$$

e.g. friction
normal force
torque due to friction
etc.

§. Lagragian EoM is equivalent to Newton's eq.

Consider 1D with $U(x)$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$x = q$$

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{x}}} = m\ddot{x}$$

↳ The partial means partial w.r.t. independent variables $x, \dot{x}, t$

implicitly dependent on t → x, $\dot{x}$

any explicit time dependence in L → t

$$\frac{\overset{\frown}{\partial} L}{\partial x} = -\frac{\partial U}{\partial x} = F(x)$$

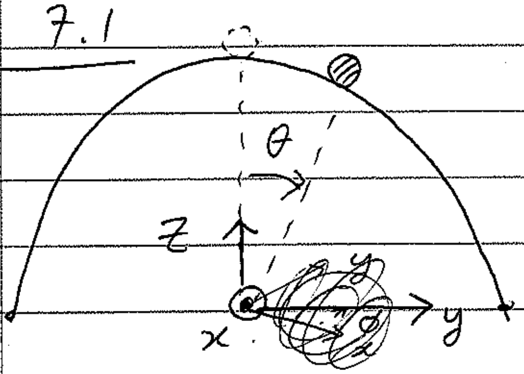
$$\therefore \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \text{means}$$

$$m\ddot{x} = F(x)$$

For a more general discussion about the equivalence, read 7.6 of the book.

↗ Do it w/ Ex 7.10

* Ex. 7.1



$$x^2+y^2+z^2=R^2$$

$$z \geq 0$$

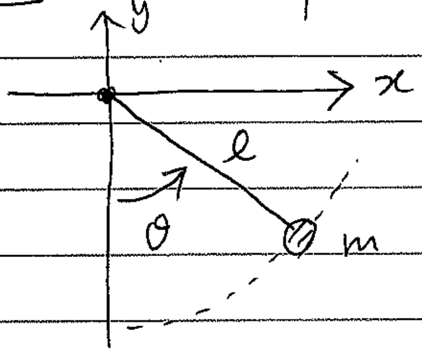
Generalized coordinates?

x, y ok

θ, ϕ better?

Do it

* Ex 7.2 Simple pendulum



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

no radial motion. only tangential

$$U = -mgl\cos\theta$$

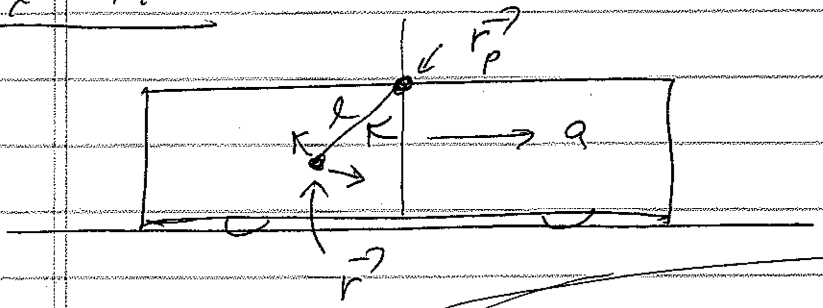
$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

E-L eq.

$$\underbrace{ml^2\ddot{\theta}}_{I\alpha \left(\leftarrow \frac{dL}{dt}\right)} = \underbrace{-mgl\sin\theta}_{\text{torque}}$$

Explit t-dependence in L.

Ex 7.6



$$\vec{r}_p = (\cancel{v_0 t} + \tfrac{1}{2}at^2)\,\hat{x}$$

$$\vec{r} = (\cancel{v_0 t} + \tfrac{1}{2}at^2 + l\,\cancel{\cos\theta}\,\sin\theta)\,\hat{x}$$
$$\cancel{+} - l\cos\theta\;\hat{y}$$

ok to ignore it to start with. But will keep it just to show that it does not matter.

$$\vec{v} = (v_0 + at + l\dot\theta\cos\theta)\,\hat{x} + l\dot\theta\sin\theta\;\hat{y}$$

$$T = \tfrac{1}{2}m\left[(v_0 + at + l\dot\theta c_\theta)^2 + (l\dot\theta s_\theta)^2\right]$$

$$U = -mgl\,c_\theta$$

$$L = \tfrac{1}{2}m\left[(v_0 + at + l\dot\theta c_\theta)^2 + (l\dot\theta s_\theta)^2\right] + mgl\,c_\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{d}{dt}\left\{\underline{m\left[(v_0 + at + l\dot\theta c_\theta)\,l c_\theta + l\dot\theta s_\theta \cdot l s_\theta\right]}\right\}$$

$$\frac{\partial L}{\partial\theta} = -m(v_0 + at)\,l\dot\theta\,s_\theta - mgl\,s_\theta$$

$$\rightarrow \frac{d}{dt}\left[m\left\{(v_0 + at)\,l c_\theta + l^2\dot\theta\right\}\right]$$

$$= m\left(a l c_\theta \cancel{+} - (v_0 + at)\,l s_\theta\dot\theta + l^2\ddot\theta\right)$$

$$l^2\ddot\theta = -\cancel{m}gl\,s_\theta - \cancel{m}al\,c_\theta$$

$$\underset{\sim\sim\sim}{\tan\theta_\varepsilon = \frac{-a}{g}}$$

$$\theta \Rightarrow \cancel{\theta}\,\phi + \theta_\varepsilon$$

$\theta = \theta_\varepsilon + \phi$ $\quad$ ϕ = small angle

(in class ⇒ $\theta_{eq} + \delta$)

$$l\ddot{\phi} = -g\, s_{\phi+\theta_\varepsilon} - a\, c_{\phi+\theta_\varepsilon} \qquad t_{\theta_\varepsilon} = -\frac{a}{g}$$

$$s_\phi c_{\theta_\varepsilon} + \cancel{c_\phi s_{\theta_\varepsilon}} \qquad \cancel{c_\phi c_{\theta_\varepsilon}} - s_\phi s_{\theta_\varepsilon}$$

$$\approx -g\, c_{\theta_\varepsilon}\phi + a\, s_{\theta_\varepsilon}\phi \qquad c_{\theta_\varepsilon} = \frac{g}{\sqrt{a^2+g^2}}$$

$$\boxed{l\ddot{\phi} = -\sqrt{g^2+a^2}\,\phi} \qquad s_{\theta_\varepsilon} = -\frac{a}{\sqrt{a^2+g^2}}$$

<u>SHO!</u>

$$\omega_0^2 = \frac{\sqrt{g^2+a^2}}{l}$$