Notes for Lecture 7

Driven SHO

7.1 Particular solution, complementary function

Suppose that a force, $F(t)$, drives a SHO.

$$\begin{aligned} m\ddot{x} + b\dot{x} + kx &= F(t) \\ \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= f(t) && \qquad f(t) \overset{def}{=} F(t)/m \\ Lx &= f(t) && \qquad L \overset{def}{=} \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2 \text{ is a } \textit{linear operator} \text{ acting on } x(t). \end{aligned}$$

Now, we have a so-called **"inhomogeneous" differential equation**, or a **differential equation with a "source."** The $f(t)$ function is the source term.

No matter, we should remember that the general solution to this ODE is the one that has two integration constants (2 times 1 as the degree of freedom is 1). If we manage to find such a solution, then by the uniqueness of the solution of Newton's equation, that is *the* general solution.

How do we find them? The answer is that we already know a lot about the above equation in its homogeneous form ($f(t) = 0$). The general solution for the homogeneous equation is called a **complementary function**, $x_c(t)$. Note that the complementary solution already contains two integration constants to adjust to any initial condition specified. What does this mean? **All we have to do is then to find one particular solution to the above equation.** Call that particular solution $x_p(t)$. This particular solution should not, and need not, have any integration constant.[1]

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[1] Should you be writing the particular solution with some extra integration constants, that means

Assume that we have found both $x_c(t)$ and $x_p(t)$. Then, the proof that $x(t) = x_c(t) + x_p(t)$ is the general solution is pretty simple, since L, as defined above, is a linear operator.

$$L(x_c + x_p) \;=\; Lx_c + Lx_p \;=\; 0 + f(t) \;=\; f(t) \qquad\qquad \text{QED.}$$

Let me remind you **what it means that L is a linear operator**. It means that for any numbers a, b and any functions $x_1(t), x_2(t)$,

$$L(ax_1(t) + bx_2(t)) \;=\; aLx_1(t) + bLx_2(t)$$

(see pages 9, 14, and 4 of lecture note 1, also). For the current SHO problem, $L = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$ is definitely a linear operator (cf. homework), due basically to the distributive rule of the differential operator $d(f+g)/dt = df/dt + dg/dt$.

A general approach that will give *any* particular solution x_p will be discussed in the next lecture. Here, we use a more elementary approach. For either approach, though, the following principle forms the foundation.

7.2 Superposition principle

A **linear system** displays this important principle. It means the following.

Suppose we have a system defined by the following equation

$$Lx(t) \;=\; f(t)$$

where L is a **linear operator** on function $x(t)$, and $f(t)$ is a source term for this equation. Further, suppose that one can divide the source term into two terms, $f(t) = f_1(t) + f_2(t)$, and that

$$Lx_1(t) \;=\; f_1(t) \;\text{ and }\; Lx_2(t) \;=\; f_2(t)$$

Then, the superposition principle means that $x_1(t) + x_2(t)$ is the solution for the original equation.

$$L(x_1(t) + x_2(t)) \;=\; f_1(t) + f_2(t) \;=\; f(t)$$

This is easy to prove since $L(x_1 + x_2) = Lx_1 + Lx_2$, due to the linearity of the operator L.

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you are repeating some part of the complimentary function in the particular solution. That is of course redundant.

You will note that we made use of this linear property already! Indeed, we already used the superposition principle! We just did not use this fancy name. For example, the solution for the homogeneous ODE was written as a linear combination of two independent solutions. This was the superposition principle, with $f_1 = f_2 = 0$. When we discussed the general solution for the driven SHO problem in the last section, that was the superposition principle, with $f_1 = 0$ (for the complementary solution) and $f_2 = f(t)$ (for the particular solution).

Notice that the superposition principle can be immediately extended to *any number of components* into which $f(t)$ can be decomposed: $f(t) = \sum_i f_i(t)$ or $f(t) = \int d\alpha \tilde{f}(\alpha, t)$. If each component gives rise to a certain solution ($f_i(t) \to x_i(t)$, $\tilde{f}(\alpha, t) \to \tilde{x}(\alpha, t)$), then the solution to the problem is

$$\begin{aligned} x(t) &= \sum_i x_i(t) & \text{for } f(t) = \sum_i f_i(t) \\ x(t) &= \int d\alpha\, \tilde{x}(\alpha, t) & \text{for } f(t) = \int d\alpha \tilde{f}(\alpha, t) \end{aligned}$$

The functions $f_i(t)$ or $\tilde{f}(\alpha, t)$ can be any functions. Typical examples are when these are Fourier expansions of the original function.[2]

Physically the superposition principle means that each solution $x_i(t)$ or $\tilde{x}(\alpha, t)$ remains intact even when it is combined with all other solutions. Individual solutions "just add up." Note that, in this addition, what is added is the amplitude ($x_i(t)$ or $\tilde{x}(\alpha, t)$), not the intensity ($|x_i(t)|^2$ or $|\tilde{x}(\alpha, t)|^2$). This is the essential feature of the superposition, important for understanding the "interference phenomena" for light and other waves.

Indeed, the superposition property is an essential property of waves, as opposed to particles. Here, we are dealing with a SHO, which would not seem like much of a wave just yet. I mean, it is not a traveling wave.[3] It can however be thought of as a localized standing wave. We will deal more extensively with waves, near the end of this course.

As you can see, the superposition principle is a great property. It means that when an arbitrary form of a driving force $f(t)$ is given, then one can first decompose $f(t)$ to a sum (or, an integral) of convenient components, solve the problem for each component force, and then sum (or integrate) all solutions!

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[2] Note that in this notation, f_i and $\tilde{f}$, if Fourier components, would include sinusoidal functions and multiplicative constants.

[3] If many SHOs are connected to each other, then we will have a traveling wave, as we will see later.

For instance, any "piecewise continuous" periodic function[4]can be written in a Fourier series.[5] Furthermore, any integrable function can be expressed as a Fourier integral. It then follows that if the response of a linear system to a force of a single sinusoid with an arbitrary angular frequency is known, then we know the response of the system to an arbitrary form of force.[6]

So, this is the reason why we solve the driven SHO problem with a sinusoidal driving force in the next section.

Before we do that, let us come back to the physics point of view of the superposition. We mentioned that the principle of superposition means that each solution $x_i(t)$ (or $\tilde{x}(\alpha, t)$) "remains intact" when they are put together. Put another way, this means that if the principle of superposition breaks down then when those solutions are put together something new happens. Indeed, if a non-linear interaction is included, then some new behaviors occur, as we will see when we discuss non-linear oscillations.

7.3 Driven SHO, a sinusoidal force

So, consider a sinusoidal driving force $F(t) = F_0 \cos(\omega t)$, *without loss of genarality*, as explained in the last section. Let $A = F_0/m$, and then the equation to solve becomes:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x \;=\; A\cos(\omega t)$$

. . . Go to the complex world (I mean complex plane).
Turn this equation into

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x \;=\; A\exp(i\omega t)$$

We solve for a *complex* $x(t)$ and then, at the end, can take the real part of it, as the actual EOM is the real part of the complex equation that we just set up. Assume

$$x(t) \;=\; C\exp(i\omega t)$$

Plugging this in, we get

$$C(-\omega^2 + 2\beta\omega i + \omega_0^2)\exp(i\omega t) \;=\; A\exp(i\omega t)$$

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[4] "Piecewise continuous" means that except at a finite number of points in any arbitrary finite interval, the function is continuous.

[5] Please familiarize yourself with Example 3.6 of the textbook. I won't quite require that you master it at this point, but you should understand the qualitative idea.

[6] Assuming that we can do the Fourier transform and the inverse-Fourier transform.

Since this equation should hold at any time, we get

$$C(-\omega^2 + 2\beta\omega i + \omega_0^2) \;=\; A$$

$$C \;=\; \frac{A}{\omega_0^2 - \omega^2 + 2\beta\omega i}$$

The magnitude of C is (call it D: $D \overset{def}{=} |C|$),

$$\boxed{D \;=\; \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}}$$

As ω increases from 0, C goes into the 4th quadrant and then the 3rd quadrant, and so it makes sense to define its phase as $-\delta$. It is given by, from the above equation of C,

$$\boxed{\delta \;=\; \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)}$$

Then,

$$x(t) \;=\; D\exp(-i\delta)\exp(i\omega t)$$

. . . Come back to the real world (I mean real axis).
Taking the real part, we get

$$\boxed{x(t) \;=\; D\cos(\omega t - \delta)}$$

with D and δ as given above. δ is the "**phase shift/lag**."

Before we go on further, let us note that the full solution is of the form

$$x_c(t) + x_p(t)$$

What we obtained just now is $x_p(t)$. We already know what $x_c(t)$ is from the previous lecture (the solution discussed in the damped SHO section). When a finite damping is present, $x_c(t)$ is always damped with the damping constant given by β or $\beta - \sqrt{\beta^2 - \omega_0^2}$ (over-damped). This means that if we wait for time $\gg$ 1/damping constant, $x_c(t)$ is negligibly small. The solution that applies at such a large value of time is called the **steady state solution**. The solution that applies to the initial time when both $x_c(t)$ and $x_p(t)$ are appreciable is called the **transient solution.** Here, we will discuss the steady state solution only, that is, $x_p(t)$ only.

Let us analyze this phase lag/shift function $\delta(\omega)$ a bit.

$$\delta \approx 2\beta\omega/\omega_0^2 \qquad\qquad \text{when } \omega \approx 0$$

Positive slope. The slope increases as damping increases.

When $\omega \approx \omega_0$, $\frac{2\beta\omega}{\omega_0^2-\omega^2} \approx \frac{2\beta\omega_0(1+\eta)}{-2\omega_0^2\eta} \approx -2\beta/(\omega_0\eta)$, where $\eta \overset{def}{=} (\omega-\omega_0)/\omega_0$. $\tan\delta \approx -2\beta/\eta$. Using $\tan\delta \approx \tan(\pi/2+\tilde{\delta}) \approx -\cot(\tilde{\delta}) \approx -1/\tilde{\delta}$, where $\tilde{\delta} \overset{def}{=} \delta - \pi/2$, we get

$$\begin{aligned} \delta &\approx \frac{\pi}{2} + \frac{\omega_0\eta}{2\beta} \qquad\qquad \text{when } \omega \approx \omega_0 \\ &= \frac{\pi}{2} + \frac{\omega-\omega_0}{2\beta} \end{aligned}$$

$\delta = \pi/2$ at $\omega = \omega_0$, and the slope there goes to infinity as $\beta \to 0$. When $\beta \to 0$, $\delta = 0$ if $\omega < \omega_0$ and π if $\omega > \omega_0$, and so this infinite slope is reasonable.

OK, that is enough analysis for the phase lag/shift. You should compare Figure 3-16(b) of the textbook with this analysis.

How do we understand $\pi/2$ phase lag at $\omega = \omega_0$? If you are good at pumping a swing, convince yourself (or not??) that this is consistent, that is, you apply torque exactly the quarter cycle before the amplitude becomes maximum (or not??). Note that when $\delta = \pi/2$, the power is delivered to the system by the external force in the most optimum way, since $x = D\cos(\omega_0 t - \pi/2) = D\sin(\omega_0 t)$ and thus $v = D\omega_0\cos(\omega_0 t)$. The power $= Fv$, and F and v are exactly in phase, both behaving as $\cos(\omega_0 t)$, when $\omega = \omega_0$.

Let us analyze the function $D(\omega)$ a bit as well. It is recognized immediately that if $\beta = 0$, then $D \to \infty$ at $\omega = \omega_0$. Thus, D in general is expected to have a peak structure around the natural frequency of the system. This is the so-called **resonance**. Near $\omega \approx 0$, $D \approx A/\omega_0$. The maximum of D is obtained by putting $dD/d\omega = 0$, which means $2(\omega^2-\omega_0^2)2\omega + 8\beta^2\omega = 0$, which means $\omega = \sqrt{\omega_0^2 - 2\beta^2}$. This is the so-called **amplitude resonance frequency**, ω_R.

$$\boxed{\omega_R = \sqrt{\omega_0^2 - 2\beta^2}}$$

$\omega_R < \omega_1 = \sqrt{\omega_0^2 - \beta^2} < \omega_0$, where ω_1 is the oscillation frequency with under-damping.

If β is small, then $\omega_R \approx \omega_0$. Near the peak, consider the *intensity* profile: $D^2 \approx \frac{A^2}{(\omega^2-\omega_R^2)^2+(2\beta\omega_R)^2} \approx \frac{A^2}{(2\omega_R)^2[(\omega-\omega_R)^2+\beta^2]}$. This is the so-called Lorentzian line shape, centered at ω_R with the full width half maximum (FWHM), 2β. For this reason, it is customary to define the Q factor (quality factor):

$$\boxed{Q \overset{def}{=} \frac{\omega_R}{2\beta}}$$

As such, Q defines how sharp the resonance behavior is, as ω is swept. Poor quality oscillator with a large damping has a poor resonance characteristics. For instance, for a critically damped or a over-damped oscillator, the resonance characteristics will be very poor. High Q factor is required for precise measurements.

The resonance frequencies for the amplitude and the potential energy are identical, since the potential energy is proportional to the amplitude squared. However, the kinetic energy of the SHO is not necessarily maximized at the same frequency. As

$$x \propto \frac{1}{\sqrt{(\omega^2-\omega_0^2)^2+4\omega^2\beta^2}}\cos(\omega t-\delta)$$

it follows that

$$\dot{x} \propto \frac{-\omega}{\sqrt{(\omega^2-\omega_0^2)^2+4\omega^2\beta^2}}\sin(\omega t-\delta)$$

$$T \propto \dot{x}^2 \propto \frac{\omega^2}{(\omega^2-\omega_0^2)^2+4\omega^2\beta^2}\sin^2(\omega t-\delta)$$

Let us use **the following *very important* result**

$$\boxed{\frac{1}{T}\int_{t_0}^{t_0+T} dt\,\sin^2(\omega t-\delta) = \frac{1}{2} \qquad \text{where } T \overset{def}{=} 2\pi/\omega \text{ and } t_0 \text{ is any real number}}$$

Then, the average kinetic energy $\langle T\rangle$ over the period T is given by

$$\langle T\rangle \propto \frac{\omega^2}{(\omega^2-\omega_0^2)^2+4\omega^2\beta^2}\,\frac{1}{2}$$

Taking the ω-derivative, $d\langle T\rangle/d\omega \propto \omega[(\omega^2-\omega_0^2)^2+4\omega^2\beta^2]-\omega^2[(\omega^2-\omega_0^2)2\omega+4\omega\beta^2] = \omega(-\omega^4+\omega_0^4)$. So, $\langle T\rangle$ has a maximum at $\omega_E = \omega_0$.[7] be the **kinetic energy resonance frequency**, occurring exactly at the natural frequency of the system, different from the amplitude/potential-energy resonance frequency $= \omega_R = \sqrt{\omega_0^2 - 2\beta^2}$.

7.4 Examples 3.4 and 3.5

There is an exact analogy between the circuit and the mechanical system.

$$L\frac{dI}{dt} + \frac{Q}{C} + RI \;=\; \mathcal{E}$$

Noting that $I = \dot{Q}$, the circuit equation becomes

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q \;=\; \mathcal{E}$$

The following correspondence can then be noted.

Circuit	Mechanical
Q	x
L	m
R	b
$1/C$	k
$\mathcal{E}$	F

Example 3.4 describes a hanging mass-spring system with an equivalent LC circuit. (Left for reading.)

Example 3.5 is an RLC resonant circuit, driven by an emf $\mathcal{E} = E_0\cos\omega t$ (correct textbook). The problem is to find the resonance frequency to maximize V_L. The voltage $V_L \;=\; L\ddot{Q}$. So, this is equivalent to maximizing the acceleration.

$$\omega_0^2 \;=\; k/m \to 1/(LC)$$
$$\beta \;=\; b/(2m) \to R/(2L)$$
$$A \;=\; F_0/m \to E_0/L$$

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[7] Here, we just showed that $\langle T\rangle$ has a unique extremum at $\omega = \omega_E$ for $\omega >= 0$. How do we know it is maximum? If you are not sure, do NOT take the 2nd derivative. That would be too complicated. Rather, note that $\langle T\rangle \to 0$ when $\omega \to 0$ or ∞, while $\langle T\rangle =$ finite at $\omega_E = \omega_0$. Since ω_E is the only extremum point between $\omega = 0$ and $\omega = \infty$, it must be a maximum.

$$
\begin{aligned}
Q &= \frac{E_0}{\sqrt{(L\omega^2 - 1/C)^2 + R^2\omega^2}} \cos(\omega t - \delta) \\
I &= \frac{-E_0 \sin(\omega t - \delta)}{\sqrt{(L\omega - 1/(C\omega))^2 + R^2}} \\
V_L &= L dI/dt \ = \ \frac{-E_0 \omega \cos(\omega t - \delta)}{\sqrt{(L\omega - 1/(C\omega))^2 + R^2}} \\
&\overset{def}{=} V(\omega) \cos(\omega t - \delta)
\end{aligned}
$$

To maximize $V(\omega)$, note that $d|V(\omega)|^2/d\omega \propto 2\omega((L\omega - 1/(C\omega))^2 + R^2) - \omega^2 2(L\omega - 1/(C\omega))(L + 1/(C\omega^2)) \propto (LC\omega^2 - 1)^2 + R^2C^2\omega^2 - (LC\omega^2 - 1)(LC\omega^2 + 1) \propto (R^2C^2 - 2LC)\omega^2 + 2$. Putting this to be 0, we get

$$
\omega_{max} = \frac{1}{\sqrt{LC - \frac{R^2C^2}{2}}}
$$

This is the frequency at which the acceleration is maximized. It is different from $\omega_E = \omega_0 = 1/\sqrt{LC}$ and the amplitude resonant frequency $\omega_R = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{1/LC - R^2/(2L^2)}$.