Notes for Lecture 6

Small Oscillations

We consider small oscillations around a stable equilibrium point. We assume that the second derivative of the potential is non-zero. The resulting motion is a simple harmonic motion (SHM), an extremely important kind of motion. It is a ubiquitous phenomenon for stable physical systems.

6.1 Simple harmonic oscillator

6.1.1 Free SHO

We will do a 1D case, here. Multi-dimensional cases and coupled oscillators will be picked up later.

We consider Hooke's law potential and force.

$$U(x) = \frac{1}{2}kx^2$$

$$F(x) = -kx$$

Note that this potential energy has no upper limit, and so we can have only bound motions. Any bound motion in 1D with conservative forces alone is a periodic motion, as discussed in the last lecture. The motion is actually a simple harmonic motion, a particular kind of periodic motion.

E is conserved, and $E \geq 0$ is required, since $U \geq 0$ and $E \geq U$.

(This way of obtaining the solution is demonstrated here, but was not discussed during the lecture.) The solution, according to the last lecture, is (choosing the negative sign for slight convenience),

$$\begin{aligned}
t &= -\int dx \sqrt{\frac{m}{2(E-\frac{1}{2}kx^2)}} \\
&= -\frac{1}{\omega_0}\int dx \frac{1}{\sqrt{A^2-x^2}} \qquad \omega_0 \stackrel{def}{=} \sqrt{k/m},\ A \stackrel{def}{=} \sqrt{2E/k} \\
&= \frac{1}{\omega_0}(\cos^{-1}(x/A) - \cos^{-1}(x_0/A)) \\
&= \frac{1}{\omega_0}(\cos^{-1}(x/A) - \theta_0)
\end{aligned}$$

Therefore, the solution is

$$\begin{aligned}
x &= A\cos(\omega_0 t + \theta_0) \\
v &= -A\omega_0 \sin(\omega_0 t + \theta_0)
\end{aligned}$$

You will notice that this solution is exactly the same form as the solution for the circular motion (last lecture, the last topic), if we make the substitution $\omega_0 \to \omega_c$, $A \to r$, and $-v/\omega_0 \to y$. How so? First of all, note that the energy conservation equation is $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$ or

$$x^2 + (v/\omega_0)^2 = A^2$$

This is a circle equation in the (X, Y) space, if we take $X = x$ and $Y = -v/\omega_0$.

Also, the EOM is of identical nature. The equation of motion for the current problem is $\ddot{x} = -\frac{k}{m}x$ or

$$\ddot{x} = -\omega_0^2 x$$

The equation of motion in the (X, Y) space is

$$\begin{aligned}
\dot{X} &= \dot{x} = v = -\omega_0 Y \\
\dot{Y} &= -\dot{v}/\omega_0 = \omega_0 X
\end{aligned}$$

If we differentiate one more time, we get

$$\begin{aligned}
\ddot{X} &= -\omega_0 \dot{Y} \\
\ddot{Y} &= \omega_0 \dot{X}
\end{aligned}$$

which has exactly the same form as before ($a_x = -\omega_c v_y$ and $a_y = \omega_c v_x$; see the last lecture).

The motion of a simple harmonic oscillator with the coordinate x corresponds to a circular motion in the $(x, -v/\omega_0)$ space.

Angular frequency $\omega_0 = \sqrt{k/m}$. (**Natural frequency**)

Amplitude A

Phase $\theta = \omega_0 t + \theta_0$

Period $T = 2\pi/\omega_0$

Frequency $\nu = 1/T = \omega_0/2\pi$

The 2D SHO (section 3.3) is left for reading (Lissajous curve).

Couple of comments. First, the "amplitude" is defined as A, but occasionally, the whole thing $x = A\cos(\omega t + \theta_0)$ is also called the amplitude. Then, A might be called the "maximum" amplitude, if more clarity is required. A as the amplitude distinguishes it from the phase (θ). x as the amplitude distinguishes it from the intensity ($|x|^2$). Second, when it may cause confusion, T as the symbol for the period may be replaced by τ.

6.1.2 Example 3.1

Physical pendulum. Sphere with a pivot point on its surface.

Rotational inertia $= mR^2 + \frac{2}{5}mR^2$ (parallel axis theorem).

Torque $= mgR\sin\theta \approx mgR\theta$ for small angle.

Newton's equation: $\dot{L} = I\dot{\omega} = I\dot{\dot{\theta}} = I\ddot{\theta} =$ torque $= -mgR\theta$ ($-$ sign means that the torque is opposite to the angular displacement in direction; in Figure 3-1 of textbook, the angular displacement vector comes out of the paper, while the torque goes into the paper, according to the right screw rule.).

$$\ddot{\theta} = -\frac{5g}{7R}\theta$$

Thus,

$$\omega_0 = \sqrt{\frac{5g}{7R}}$$

The reason why this frequency is independent of mass is because of the equivalence principle.

In general, for a physical pendulum with a small angular amplitude, we get

$$\omega_0 = \sqrt{\frac{mgR}{I}}$$

This of course applies to the **simple pendulum** case, for which $I = ml^2$, $l = R$ (length of the pendulum),

$$\omega_0 = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

which is the well-known result.

6.2 Phase space

The phase space is defined as space with coordinates (x, p), where p is the momentum. This is for 1D and 1 particle. In the most general case, the phase space is defined as space with coordinates $(x_1, x_2, \cdots, x_M, p_1, p_2, \cdots, p_M)$. Here, M is the degrees of freedom.

What we just showed is then a SHM (simple harmonic motion) is a circular motion in phase space, assuming that the p axis is scaled properly by $-1/(m\omega_0)$.

Keep in mind the following properties of the phase space.

- Specifying the initial condition of a system is equivalent to specifying a point in the phase space at time 0.
- The time evolution of that point is completely determined by Newton's laws.
- The motion of a particle, or a system of particles, is represented as a path in the phase space.
- Suppose you prepare an otherwise identical system with two different initial conditions. As time goes on, the two points will move in phase space. Those two points can never occupy the same point in phase space at the same time.
- Such two paths either are identical (if the two initial conditions are related to each other by a time offset), or never intersect each other.

Phase space is important, for example, for statistical mechanics and for understanding certain electron optical elements.

6.3 Damped SHO

A more realistic oscillator has damping/friction. Model it as $-bv$ ($b > 0$). EOM:

$$\begin{aligned} m\ddot{x} &= -kx - bv \\ \ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= 0 \qquad\qquad \omega_0 = \sqrt{k/m},\ \beta \stackrel{def}{=} b/(2m) \end{aligned}$$

Here, β is called the **damping parameter**.

Damping means no energy conservation, and so we cannot use the method that we used for the free SHO, in this case.

What is the general method to solve this important equation?

Note that the EOM is linear. What does it mean? Define

$$L \stackrel{def}{=} \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

L is an "operator" acting on $x(t)$. The EOM can be written as

$$Lx(t) = 0$$

Due to L being a *linear*[1] operator, if we manage to find two independent solutions, $x_1(t)$ and $x_2(t)$ to $Lx(t) = 0$, then we can form a linear combination with two constants, $A_1x_1 + A_2x_2$, which would be the general solution to the problem. The art is to find x_1 and x_2. While a more systematic theory such as the Sturm-Liouville theory or a series expansion method exists, the current problem is a well-known problem and one has to be very familiar with how to obtain solutions easily. We try $x(t) = \exp(\alpha t)$ into the EOM. We get

$$(\alpha^2 + 2\beta\alpha + \omega_0^2)e^{\alpha t} = 0$$

In order for this to be true for any t value, we must require that

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0$$

which means

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

So, the general solution is

$$x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}\, t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}\, t\right)\right]$$

[1] $L(A_1x_1 + A_2x_2) = A_1Lx_1 + A_2Lx_2$ for any numbers A_1, A_2 and functions x_1, x_2

if $\omega_0^2 \neq \beta^2$. This corresponds to the case of underdamping ($\beta^2 < \omega_0^2$) or overdamping ($\beta^2 > \omega_0^2$).

If $\omega_0^2 = \beta^2$ (critical damping), then there is only one α value ($-\beta$), and so the above solution reduces to one function ($\exp(-\beta x)$) only. So we have to find another solution. In this case, by direct substitution, $te^{-\beta t}$ is shown to be a solution, and so the general solution is (for $\omega_0^2 = \beta^2$):

$$x(t) \ = \ e^{-\beta t}(A_1 + A_2\, t)$$

How could we have guessed that the other solution is $te^{-\beta t}$? You can take a more systematic point of view as follows. We write down the solution as $g(t)e^{-\beta t}$ and then figure out what kind of equation $g(t)$ must satisfy: it is $\ddot{g}(t) = 0$ (left as exercise).

6.3.1 Underdamping

For an underdamped SHO ($\omega_0 > \beta$), define

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

Then,

$$x(t) \ = \ e^{-\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]$$

Well, why did we get a complex quantity here? The answer is that the above EOM is perfectly happy with a complex solution. We are not necessarily unhappy with it, but should know that the physical solution must be real. We must require that the solution is a real number. That means $x^*(t) = e^{-\beta t}\left[A_1^* e^{-i\omega_1 t} + A_2^* e^{i\omega_1 t}\right] = x(t)$. This is possible only if $A_1 = A_2^*$. Write $A_1 = Ae^{i\theta_0}/2$, where $A > 0$. Then,

$$x(t) = e^{-\beta t} A\left[e^{i(\omega_1 t + \theta_0)} + e^{-i(\omega_1 t + \theta_0)}\right]/2$$

Thus, we get

$$x(t) \ = \ Ae^{-\beta t}\cos(\omega_1 t + \theta_0)$$

- As β becomes small, then the solution approaches the correct limit (free SHO), since $\beta \to 0$ and $\omega_1 \to \omega_0$:

  $$x(t) \ = \ A\cos(\omega_0 t + \theta_0)$$

  In the current context, it means that the general solution for

  $$\ddot{x} = -\omega_0^2 x$$

  can be written as

  $$x(t) \ = \ A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t}$$

  This **complex solution** for the free SHO is worth remembering.

- When β is small, the exponential decay term will decay very slowly, while the cosine term will oscillate rapidly.
- An underdamped SHO is just like a free SHO except that the amplitude is damped by the time scale β. The oscillation is defined by an "envelope function" $Ae^{-\beta t}$.
- As β approaches ω_0, the oscillation becomes very very slow. $\omega_1 \to 0$ and T (period) $\to \infty$!

6.3.2 Overdamping

For an overdamped SHO ($\omega_0 < \beta$), define

$$\gamma = \sqrt{\beta^2 - \omega_0^2}$$

then the solution becomes

$$x(t) \; = \; e^{-\beta t} \left[A_1 e^{\gamma t} + A_2 e^{-\gamma t} \right]$$

Note that $0 < \gamma < \beta$, so that $\beta - \gamma > 0$. The two decay constants are, now, $\beta + \gamma$ (fast decay; the 2nd term) and $\beta - \gamma$ (slow decay; the first term).

6.3.3 Critical damping

For a critically damped SHO ($\omega_0 = \beta$), we already wrote down the solution above:

$$x(t) \; = \; e^{-\beta t}(A_1 + A_2 \, t)$$

This is the case when the decay behavior is governed by one decay constant β. Clearly the overdamping would not be efficient in dampening a vibration than the critical damping, due to the slow decay term existing in the overdamped SHO solution. Indeed, the critical damping is the best way to stop vibrations the most quickly.

6.3.4 Example 3.3

Damped pendulum. $F_{res} = 2m\sqrt{g/l}\; l\dot{\theta}$. Torque due to this $= 2m\omega_0 l^2 \dot{\theta}$.

EOM:

$$ml^2 \ddot{\theta} \; = \; -mgl\theta - 2m\omega_0 l^2 \dot{\theta}$$

$$\ddot{\theta} + 2\omega_0\dot{\theta} + \omega_0^2\theta = 0$$

This is a critically damped oscillator.

$$\theta(t) = (A + Bt)\exp(-\beta t)$$

The initial condition is such that $A = \alpha$.

$$\dot{\theta}(t) = (B - \beta A - \beta Bt)\exp(-\beta t)$$

Since $\dot{\theta}(0) = 0$, $B = \beta A = \alpha\beta$.

$$\theta(t) = \alpha(1 + \beta t)\exp(-\beta t)$$

$$\dot{\theta}(t) = -\alpha\beta^2 t\exp(-\beta t)$$

where $\beta = \omega_0$. The EOM can be re-written as (using the work energy theorem $\tau d\theta = -d(\frac{1}{2}I\dot{\omega}^2)$ or the differential calculation trick):

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} + 2\omega_0\dot{\theta} + \omega_0^2\theta \ = \ 0$$

From which we get

$$\frac{d\dot{\theta}}{d\theta} \ = \ -2\omega_0 - \omega_0^2\frac{\theta}{\dot{\theta}}$$

From the above solution, this means

$$\frac{d\dot{\theta}}{d\theta} \ = \ -2\omega_0 + \omega_0^2\frac{1 + \omega_0 t}{\omega_0^2 t}$$

where the substitution $\beta \to \omega_0$ is made. One can see that the slope of the path in the phase space is initially ∞ as the right hand side diverges at $t = 0$. This is because initially the effect of resistance is near zero, and the path is an ellipse (or a circle). As $t \to \infty$, the slope approaches $-\omega_0$. Also, it should be noted that θ never changes sign, given the current initial condition. So, we can sketch a plot like that shown Figure T3-14, from this qualitative reasoning. Or, we can simply use the computer to parametrically make such a plot.