Notes for Lecture 4

Lorentz force

4.1 Example 2.10 of text

Motion of a charge in a uniform $\vec{B}$ field.

$\vec{B} = B_0\hat{z}$. (This is a more common convention. The textbook chooses the y direction.) The Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$, where q is the electric charge of the particle. Here we consider a simple example, where $\vec{B}$ is constant.

(Also, below, I prefer using $\hat{x}, \hat{y}, \hat{z}$ instead of $\vec{i}, \vec{j}, \vec{z}$, as i, j, k are commonly used as summation indices!)

Any vector product can be calculated, if the following basic rules are noted, along with other usual properties of multiplication (associative rules, distributive rules, ...).

$$
\begin{aligned}
\vec{A} \times \vec{B} &= -\vec{B} \times \vec{A} && \textit{anti}\text{-commutative!} \\
\hat{x} \times \hat{y} &= \hat{z} \\
\hat{y} \times \hat{z} &= \hat{x} \\
\hat{z} \times \hat{x} &= \hat{y}
\end{aligned}
$$

All of the following properties can be derived from the above basic rules. Knowing these by heart will server you well.

1. $\vec{A} \times \vec{A} = 0$.

2. **Right screw/hand rule** applies. Rotate $\vec{A}$ towards $\vec{B}$ (involving the

shorted angular displacement possible). How would a right-handed screw move along its axis on such a rotation? That is the direction of $\vec{A} \times \vec{B}$.

3. $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$: $(\vec{A} \times \vec{B}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot \vec{B} = 0$.

4. $|\vec{A} \times \vec{B}|$ is the **area of the parallelogram** spanned by $\vec{A}$ and $\vec{B}$, i.e. twice the area of the triangle formed by $\vec{A}$ and $\vec{B}$. It is $AB\sin\theta$.

5. $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \hat{i} A_j B_k$, where ε_{ijk} is the Levi-Civita symbol ($\varepsilon_{123} = 1$ and it changes sign whenever two indices are swapped, i.e. permuted), and $\hat{1} \stackrel{def}{=} \hat{x}$, $\hat{2} \stackrel{def}{=} \hat{y}$, $\hat{3} \stackrel{def}{=} \hat{z}$. Here, the middle term means the determinant of the 3×3 matrix. Strictly speaking this matrix does not make sense as its elements must be numbers, not vectors. However, as *mnemonics*, this matrix serves us well.

6. In other words, $(\vec{A} \times \vec{B})_3 = A_1 B_2 - A_2 B_1$ and, by **cyclic permutations of indices**, $(\vec{A} \times \vec{B})_2 = A_3 B_1 - A_1 B_3$ and $(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$.

7. $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$. The magnitude $|(\vec{A} \times \vec{B}) \cdot \vec{C}|$ is the volume of the parallelepiped formed by these three vectors.

8. Finally, an essential identity! $\sum_{k=1}^{3} \varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

The Lorentz force is then $q\vec{v} \times B_0\hat{z} = q(v_y B_0 \hat{x} - v_x B_0 \hat{y})$. Notice that there is no force along the z direction, since $\hat{z} \times \hat{z} = 0$. So, $v_z = v_{z,0}$.

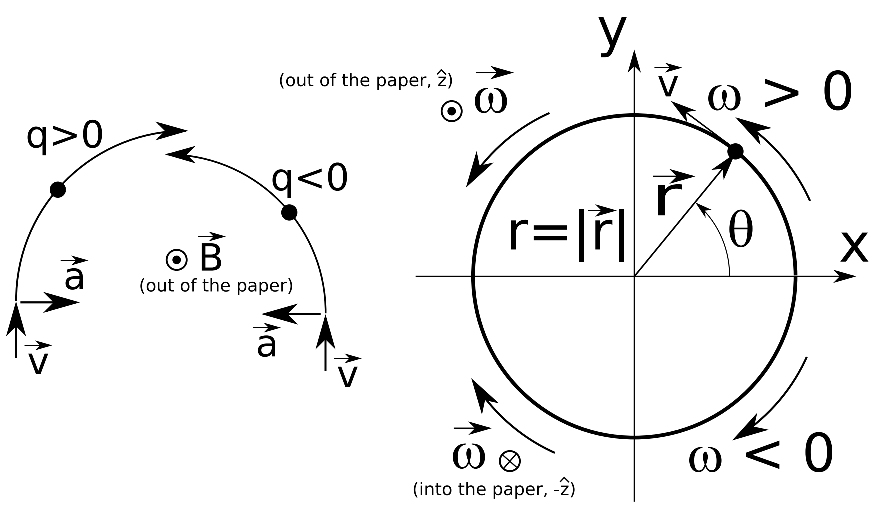
Let us use the inertial frame, which is moving at $v_{z,0}\hat{z}$. Then, there is no z motion, and we can simply deal with x and y motions only.

$$\begin{aligned} ma_x &= qv_y B_0 \\ ma_y &= -qv_x B_0 \end{aligned}$$

$$\begin{aligned} a_x &= -\omega_c v_y \qquad\qquad \omega_c \stackrel{def}{=} -qB_0/m \\ a_y &= \omega_c v_x \end{aligned}$$

$$\begin{aligned} \vec{a} &= \omega_c(-v_y \hat{x} + v_x \hat{y}) \\ \vec{a}\cdot\vec{v} &= 0 \qquad\qquad \omega_c(-v_y v_x + v_x v_y) = 0 \end{aligned}$$

Here, $\omega_c = -qB_0/m$ has the dimension of inverse time, and it is call the "**cyclotron frequency**." The motion is a **uniform circular motion**, with the angular frequency given by ω_c. How do we know this? (1) The acceleration is always perpendicular to the velocity. So, it is a centripetal acceleration. So, it is a circular motion. There is no tangential acceleration, so v will be constant. So, a uniform circular motion. (2) The magnitude of the acceleration is, $a = \sqrt{\omega_c^2 v_x^2 + \omega_c^2 v_y^2} = |\omega_c| v$. This is precisely the uniform circular motion relation that you learned in an elementary course, with the angular frequency $|\omega_c| = 2\pi/T$ where T is the period.[1]



These arguments are quite satisfactory, but let us do some more work here, now that we know precisely what it means to solve a Newton's equation. Notice that we have a 2D problem of one particle, so the number of integration constants (i.e. constants for specifying the initial condition; see below for more information) is four.

[1]Unfortunately, T is a versatile symbol, used for multiple purposes. Sometimes it is "time" (dimension). Sometimes it is period. Sometimes it is kinetic energy. Or, temperature. Usually the context makes the meaning unambiguous.

How many numbers do we need to characterize a uniform circular motion? 2 for the position of the center. 1 for the radius r. 1 for the speed v. 1 for the initial phase (θ_0),[2] the initial angular position. It seems that we have 5 – which is too many. But, remember that $v = 2\pi r/T$, and so r and v are physically constrained by ω_c, i.e. by B_0 and q/m. So we have 4 numbers. Good. As the origin of the coordinate system is arbitrary, let us choose the most convenient coordinate system so that the center of the circle is the origin. Then, we expect to have two integration constants in the solution.

Now, we are ready to write down the solution. By using the above coordinate system, and identifying $\theta = \omega_c t + \theta_0$, we get

$$\begin{aligned} x &= r\cos(\omega_c t + \theta_0) \\ y &= r\sin(\omega_c t + \theta_0) \end{aligned}$$

Here, r, θ_0 are two integration constants that need to be fixed by the initial condition [3] Upon taking the derivative, $\dot{x} = -\omega_c y$, and $\dot{y} = \omega_c x$. Repeating, $\ddot{x} = -\omega_c \dot{y}$, and $\ddot{x} = \omega_c \dot{x}$. This agrees with the equations that we wrote above ($a_x = -\omega_c v_y$ and $a_y = \omega_c v_x$). This ends the proof that the above solution is the general solution.

If we include the z motion, then the motion is that or a cylindrical spiral motion, a circular motion in the $x - y$ plane plus a uniform translation along the z axis.

Notice that we defined ω_c so that it is negative for the positive charge and positive for the negative charge. This is the result of our setting up the coordinate system as shown above.

Note that the angular velocity vector $\vec{\omega}$ is in general defined through a right-hand rule. We will study more details later, but, at this point, it should suffice to study the direction of $\vec{\omega}$ in the above diagram.

As you can see, the cyclotron motion can be useful to figure out what the sign of the particle's electric charge is, or what the kinetic energy of the particle is after a collision, if q/m is known.

Lastly, note that the time reversal symmetry is broken for this problem. What this means is that when the motion is played backwards (without changing the direction of $\vec{B}$: often the direction of $\vec{B}$ is thermodynamically determined and it is a given, non-reversible, or it involves a disspative process and it is non-reversible), then an

[2] The "phase" is defined as θ. And so, θ_0 is the *initial* phase.

[3] ω_c is not an integration constant here! During the lecture, I got confused by thinking that it is! Sorry. So, it is alright. Out of four integration constants, two of them got taken care of by fixing the origin, and we have two here.

impossible motion results (e.g. a positive charge doing a clock-wise rotation will results in a positive charge doing a counter-clock-wise rotation upon the timer reversal).

However, different from dissipative problems (friction, air resistance), which also break the time reversal symmetry, the mechanical energy is conserved here. The Lorentz force is perpendicular to the velocity, and so it actually **does not do any work**.