Notes for Lecture 3

Perturbation, air resistance (cont.)

3.1 Perturbation

As a general strategy to solve a real world problem, the perturbation theory is very important. Please read Appendix A (and your note taken during class), which describes the rules (page 5 and examples) and also the theory behind them (the first few pages).

Let us solve the textbook example 2.4 perturbatively, assuming that the air resistance is small.

$$
\begin{aligned}
\dot{v} &= -kv && \text{assume } -kv \text{ is small} \\
\dot{v} &\approx 0 && \text{0-th order equation; ignore the small term} \\
v &\approx v_0 && \text{0-th order solution} \\
\dot{v} &\approx -kv_0 && \text{1-st order equation; plug the 0-th order solution into the small term} \\
v &\approx v_0 - kv_0 t = v_0(1-kt) && \text{1-st order solution} \\
\dot{v} &\approx -kv_0(1-kt) && \text{2nd-order equation; plug the 1-st order solution into the small term} \\
v &\approx v_0 - v_0(kt) + \frac{1}{2}v_0(kt)^2 && \text{2nd-order solution} \\
&\cdots
\end{aligned}
$$

One thing to note in the above solution is the emergence of a dimensionless parameter kt. We can define this as our perturbation parameter, $\alpha \equiv kt$ (or, we could use $\lambda \equiv kt$; λ is a slightly more frequently used symbol for a perturbation parameter).

It is important to note that when one speaks of small or large, one always means a **dimensionless parameter** being small or large, as it is not well-defined to speak of a dimension-ful quantity as being large or small (only a unit change will make it seem arbitrarily large or small). So, like kt, **the perturbation parameter must be dimensionless**.

The solution shown above as "the first-order solution" means that the solution that is correct up to $O(\alpha)$. Similarly, the second-order solution is correct up to $O(\alpha^2)$.

Here, the first order solution $v = v_0 - \alpha v_0$ has the correction $-\alpha v_0$ to the "unperturbed solution" (i.e. the 0-th order solution) $v = v_0$. In this sense, $-\alpha v_0$ is called the **leading order correction**. In this example, the leading order correction is linear in α, i.e. $O(\alpha)$. For other problems, the first order correction might be zero, and the second order correction might be non-zero. In such a case, the leading order correction would be of 2nd order, $O(\alpha^2)$.

This problem is simple enough to see that this series will converge to, if you repeat this process indefinitely, to the exact solution $v_0 \exp(-kt)$, which is not surprising at all. This is an exceptional case due to the problem being very simple. In a complex real world problem, such a feat of obtaining an exact solution using a perturbation theory would be hard to accomplish. Often the crucial step towards making progress is to identify an essential simple model that is exactly solvable and then add other effects that can be treated perturbatively to give useful solutions.

The general importance of the perturbation method is that it is applicable to any problem, or any "normal"[1] physics problem with a small perturbation parameter. For real life problems that are unsolvable by hand and involve too many degrees of freedom to be solvable by a computer in a finite time, the perturbation method is often the only tool that we have. In general, the first thing to do is to put your problem/equation in this form

$$F(x) = \lambda G(x)$$

where x is the unknown variable to solve for, $F(x) = const.$ is an easy-to-solve equation and λ is a small perturbation parameter. Then, the perturbation solution can be obtained by recursion, starting from the zero-the order solution, and then plugging in the (n-1)-th order solution to the RHS, turning the expression $\lambda G(x)$ into a *mere number* $\lambda G(x_{s,n-1})$ where $x_{s,n-1}$ is the n-th order solution. The n-th order solution is then obtained rather easily by solving

$$F(x) = \lambda G(x_{s,n-1}) \text{ where } n = 1, 2, 3, \cdots$$

It is crucial to know that at each step of the recursion, the solution $x_{s,n}$

[1] Warning: Some problems of Nature are fundamentally "non-perturbative." Those are also very interesting problems! We will get some taste of them when we deal with non-linear dynamics.

is correct only up to $O(\lambda^n)$. The apparent solution for $F(x) = \lambda G(x_{s,n-1})$ may appear to contain high order terms. They are not just unnecessary, but also *incorrect* in general. See example A.2 for the illustration of this point. This fact that the n-th order solution is correct only up to $O(\lambda^n)$ is actually a blessing in terms of finding the solution for $F(x) = \lambda G(x_{s,n-1})$, as one can simply throw away terms that are of higher orders.

3.2 Example 2.5 of text

Vertical motion, $F = -kmv$ **plus gravity**

The force is now $F = -mg - kmv$, and $v = \dot{z}$. $z \stackrel{def}{=}$ vertical axis (up is positive).

$$
\begin{aligned}
m\dot{v} &= -mg - kmv \\
\frac{dv}{dt} &= -g - kv && \text{divide by } m\text{, use Leibniz} \\
\frac{dv}{g + kv} &= -dt && \times dt,\ /(g + kv) \\
\frac{1}{k}\ln[(g + kv)/(g + kv_0)] &= -t && \text{integ. from t=0 to t=t} \\
\frac{g + kv}{g + kv_0} &= \exp(-kt) && \text{note, } kt = \text{dimensionless} \\
\frac{v + g/k}{v_0 + g/k} &= \exp(-kt) && /k \text{ upstairs and downstars, LHS} \\
\frac{v + v_t}{v_0 + v_t} &= \exp(-kt) && v_t \stackrel{def}{=} g/k \\
v &= (v_0 + v_t)\exp(-kt) - v_t && \times \text{ and } - \\
v &= v_0 + (v_0 + v_t)[\exp(-kt) - 1] && + \text{ and } -
\end{aligned}
$$

Checks: The dimension of $v_t = g/k = (L/T^2)/(1/T) = L/T$ (velocity). (Known limit) If $k = 0$? (no air friction): $\exp(-kt) \approx 1 - kt$, we get $v = (v_0 + g/k)(1 - kt) - g/k = v_0 - gt$ (correct!). (Trend check) As t increases, $\exp(-kt)$ decreases, and so v decreases (becomes more negative). This is due to the pull of the gravity. However, due to the air resistance, v does not decrease indefinitely.

Terminal velocity As $t \to \infty$, we see that $v \to -v_t$. The **terminal speed** v_t is determined by the balance between the downward pull of the gravity and the upward resistance: $mg = mkv_t$.

Integrate v (2nd to last form) to get z:

$$z = z_0 + (v_0 + v_t)[1 - \exp(-kt)]/k - v_t t \tag{3.1}$$

Checks: Two symbols for the initial condition are z_0 and v_0. You should challenge yourself to show that you get $z = z_0 + v_0 t - gt^2/2$ in the $k \to 0$ limit [Hint: use $\exp(-kt) \approx 1 - kt + (kt)^2/2$], and that $z(t \to \infty) = const. - v_t t$.

Galilean invariance in action

Notice that, compared to the previous example, the only thing that is different in this example is the finite terminal velocity, v_t. Prove to yourself the following. In the reference frame falling with velocity $-v_t$ (which is another inertial frame, due to Galilean invariance), the solutions of this example become exactly the same (with the symbol change $z \to x$) as those of the previous example.

3.3 Example 2.6 of text

No air resistance, Projectile motion, Initial speed v_0, Angle θ

This is an elementary problem at the level of the introductory course. However, it doesn't hurt to review it. The key is that x, y motions can be tackled independently. Define the y axis to point up: gravity $= -mg\hat{y}$.

$$\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= -g \end{aligned}$$

Integrate each ODE.

$$\begin{aligned} v_x &= v_{x,0} \\ v_y &= v_{y,0} - gt \\ x &= x_0 + v_{x,0} t \\ y &= y_0 + v_{y,0} t - \frac{1}{2} g t^2 \end{aligned}$$

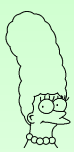
$v_{x,0} = v_0 \cos\theta \stackrel{def}{=} U$ (in the book) and $v_{y,0} = v_0 \sin\theta \stackrel{def}{=} V$ (in the book).

$$\begin{aligned} v_x &= v_0 \cos\theta \\ v_y &= v_0 \sin\theta - gt \\ x &= x_0 + v_0 t \cos\theta \\ y &= y_0 + v_0 t \sin\theta - \frac{1}{2} g t^2 \end{aligned}$$

It immediately follows that the **trajectory**, the graph (x, y), is a parabola, since x is a first order polynomial of t and y is a second order polynomial of t, and accordingly y is a second order polynomial of x.

$T \stackrel{def}{=}$ **time of flight** $\stackrel{def}{=}$ the time between the initial point to the final point. Let us take the final point to be when the projectile comes back to the original height y_0. Then, due to the obvious symmetry of the problem (see box), $T/2$ is the time when the projectile reaches the top of the trajectory.

$$T/2 = v_0 \sin\theta / g$$



Time-reversal symmetry (advanced)

What is the "symmetry" of this problem that we just mentioned as being "obvious"? It is called the **time reversal symmetry**. It sounds like fancy words, but it is not so difficult to define. Here is the definition. Say we have a specific Newton's equation to solve. Pick an arbitrary motion satisfying that equation. Now, imagine playing the "movie" of the motion backwards. Is it another *possible* solution of the same Newton's equation? If the answer is yes for any arbitrary solution, then the time-reversal symmetry is present for that Newton's equation. Convince yourself that the current example has this symmetry, but the previous two examples do not. Whenever there is "dissipation" (friction, air-resistance) or a magnetic field, the time-reversal symmetry is not present, or, we say, is *broken.*

The **range of motion**, $R \stackrel{def}{=} x(t = T) - x_0$

$$\begin{aligned} R &= \frac{2v_0^2 \sin\theta \cos\theta}{g} & \leftarrow v_x T \\ &= \frac{v_0^2 \sin 2\theta}{g} \end{aligned}$$

This gives the familiar result that the maximum range is obtained if $\theta = 45°$. This is the **optimum angle of throw** in the case when there is no air resistance.

3.4 Example 2.7 of text

Now, add air resistance $-km\vec{v}$.

The solution to this case consist of collecting all the ground work done in previous Sections (2.3, 3.2).

The x motion is that of an exponential approach to the finite range, $v_{x,0}/k$ as in Section 2.3. From Equation 2.4:

$$x = x_0 + \frac{v_{x,0}}{k}[1 - \exp(-kt)]$$

The y motion is that of Section 3.2. From Equation 3.1:

$$y = y_0 + (v_{y,0} + v_t)[1 - \exp(-kt)]/k - v_t t$$

where $v_t = g/k$.

When the air resistance is finite:

- The trajectory is no longer a parabola, as can be seen from the above two equations.
- In fact, the trajectory is no longer symmetric with respect to the point of maximum height. As v_x keeps decreasing, the trajectory becomes steeper as the mass comes down. (Think a fly ball in baseball.)
- The mechanical energy is not conserved, because it is lost at the rate $dE/dt = \vec{F}_{air} \cdot \vec{v} = -kmv^2$. Thus, when the particle is thrown up and then comes back to the same height, the speed will be smaller.
- The time-reversal symmetry is broken. This follows from the last two items.

Let us solve for the time of flight T for which $y = y_0$.

$$\begin{aligned} v_t T &= (v_{y,0} + v_t)[1 - \exp(-kT)]/k \\ \xi &= (1+\alpha)[1 - \exp(-\xi)] \qquad \xi \stackrel{def}{=} kT,\ \ \alpha \stackrel{def}{=} v_{y,0}/v_t = k v_{y,0}/g \end{aligned}$$

In general, the solution of this equation for ξ can be easily obtained using Numerical methods.

It is very instructive, though, to consider the solution in limiting cases.

1. $k \to 0$. Low air resistance. The first approximation will be the value of ξ without any air resistance: $\xi \approx 2v_{y,0}k/g = 2\alpha$, and so the solution for ξ is a small number. Thus, we can expand

$$1 - \exp(-\xi) \ = \ \xi - \frac{1}{2}\xi^2 + \frac{1}{6}\xi^3 + \cdots$$

So, we get

$$\xi \ = \ (1+\alpha)(\xi - \frac{1}{2}\xi^2 + \frac{1}{6}\xi^3 + \cdots)$$

Cancelling ξ, we get

$$1 \ = \ (1+\alpha)(1 - \frac{1}{2}\xi + \frac{1}{6}\xi^2 + \cdots)$$

Rearranging,

$$\frac{1+\alpha}{2}\xi \ = \ \alpha + (1+\alpha)\left(\frac{1}{6}\xi^2 + O(\xi^3)\right)$$

We get

$$\xi = \frac{2\alpha}{1+\alpha} + 2\left(\frac{1}{6}\xi^2 + O(\xi^3)\right)$$

Now, **this is the form to which the perturbation method can be applied.** Within the perturbation formalism (Appendix A), $F(\xi) = \xi - \frac{2\alpha}{1+\alpha}$, and $\alpha G(\xi) = 2\left(\frac{1}{6}\xi^2 + O(\xi^3)\right)$. Since ξ and α are of the same order, we expand the first term to second order as well

$$\frac{2\alpha}{1+\alpha} \ = \ 2\alpha(1 - \alpha + \cdots) = 2\alpha - 2\alpha^2 + \cdots$$

Therefore,

$$\xi \ = \ 2\alpha - 2\alpha^2 + \frac{1}{3}\xi^2 + O(\alpha^3)$$

Thus, finally, plugging in $\xi \approx 2\alpha$, the 0-th order solution, on the RHS, we get

$$\xi \ \approx \ 2\alpha - \frac{2}{3}\alpha^2$$

Or,

$$T \approx T_0(1 - \frac{1}{3}\alpha)$$

where $T_0 = 2v_{y,0}/g$ is the unperturbed time of flight. So, the time of flight is decreased relative to the "un-perturbed" value, i.e. the value for $k = 0$.

Although the above discussion is alright, there is a bit of a problem, one might say, since the "zero-th order" solution for ξ is in fact the first order in α. (However, notice that such a zero-th order solution does give the correct zero-th order solution for T.)
One can make this somewhat better formally, by doing the following. Instead of doing the above, let us do

$$F(\xi) = \alpha G(\xi)$$

where $F(\xi) = \xi$ and $\alpha G(\xi) = \frac{2\alpha}{1+\alpha} + 2\left(\frac{1}{6}\xi^2 + \cdots\right)$. Then, it follows that the zero-th order solution for ξ is 0 (which makes sense since $\xi = kT \propto k$ and so it must be linear in α or higher!), and the first order solution is 2α and then the second order solution is $2\alpha(1 - \frac{1}{3}\alpha)$ by doing the same math as above. Now, noting that ξ itself contains a factor of k, since $\xi = kT$, one can see that the first order solution for ξ gives the zero-th order solution for T, and the second order solution for ξ gives the first order solution for T (and so on).

2. $k \to \infty$. High air resistance. In this case, $\xi \to \infty$ and $\alpha \to \infty$, and so $\exp(-\xi)$ can be safely ignored, and $(1 + \alpha) \approx \alpha$. Thus, $\xi \approx \alpha$. As $\xi = kT$ and $\alpha = kv_{y,0}/g$, this means $T \approx v_{y,0}/g$. Namely,

$$T \approx T_0/2$$

In this case, T is determined by the downward motion only, to the first approximation. How is it so? Here is the explanation. The maximum height $\approx v_{y,0}/k$ (show it), and the time to reach the maximum height $\approx \ln(v_{y,0}k/g)/k$ (show it). On the way down, the terminal speed g/k is reached after time $\sim 1/k$, but the distance traveled during this time $\sim g/k^2 \ll v_{y,0}/k$, and the rest of the trip takes $\approx v_{y,0}/k/(g/k) \approx T$.
To summarize, even though the maximum height is near zero ($v_{y,0}/k$), because the terminal velocity is reached almost instantly in the downward motion, and because the terminal velocity is also very small (g/k), it takes a finite time for the downward motion ($(v_{y,0}/k)/(g/k) = v_{y,0}/g$), which is approximately equal to the total time of flight.

$k \to 0$ or $k \to \infty$, that is confusing!

If you carefully examine the above discussions, you will note that $k \to 0$ really means $\alpha = v_{y,0}/v_t \ll 1$ and $k \to \infty$ really means $\alpha \gg 1$, as we now understand in terms of a dimensionless perturbation parameter. For example, even if k is "large," say 10 sec^{-1}, if $v_{y,0}$ is "small," say 0.1 m/s, then this corresponds to a very small air resistance problem, that is, case 1 not case 2. **When a certain parameter is said to be small or large, always look for a dimensionless version of that parameter in the equation. It is that dimensionless parameter which is large or small.** In the current example, α is the dimensionless version of k. This makes sense. The only way to judge whether k is large or small is to compare it with another time scale of the problem, which is, in our current example, $g/v_{y,0}$. In this way, $\alpha = k/(g/v_{y_0})$, can be also understood as the ratio of the time scale for the air resistance and the time scale of the free fall. So, the expressions such as $k \to 0$ and $k \to \infty$ are not the best expressions to use, but they should be re-interpreted in terms of α.

From these considerations of the two limiting cases, one may expect that T gradually decreases from T_0 to $T_0/2$, as k increases from 0 to ∞. That this is indeed the case can be verified both numerically and analytically (left for your optional work).

Now, let us ask a question. Will the optimum throw angle be different from $\theta = 45°$? If so, will it be greater than $45°$ or less than it?

The qualitative answer is easy to get, given the above results.

Let us first analyze, in the case of no air resistance, what determined the optimum throw angle. Two factors.

1. The longer the time of flight, the better.
2. The larger $v_{x,0}$, the better.

These are conflicting requirements, the first $\propto v_{y,0} \propto \sin\theta$ and the second $\propto \cos\theta$. Since the range is the product of the time of flight and $v_{x,0}$, the answer is to optimize the product $\sin\theta\cos\theta$.

Now, consider air resistance. Notice that the time of flight is still $\propto v_{y,0}$ and the range of motion is still $\propto v_{x,0}$. However, the range of motion is not directly proportional to the time of flight any more. Essentially, the x motion crawls to a stop, after the time scale $1/k$ has passed. So, there is a sort of "diminishing return" for optimizing the time of flight. To make up for the diminishing return, it is necessary to increase $v_{x,0}$, in order to move the projectile farther in the initial stage. Thus, the answer is that $\theta < 45°$ is the optimum throw angle.

While this qualitative argument is precise, it does not answer the question "how much?". One can verify this argument and get a quantitative answer analytically by examining the two limits considered above (homework).