Due Nov. 10, Thursday.

Problem 1 (10 points) **Mechanical similarity.**

(a) Show that scaling the Lagrangian

$$L'(q, \dot{q}, t) = CL(q, \dot{q}, t)$$

where C is a non-zero constant, does not change physics ~~either~~, in the sense that the Hamilton's principle $\delta S' = 0$ is completely equivalent to $\delta S = 0$.

(b) Assume a power law potential energy $U = ax^n$ for a one dimensional motion. Suppose in the Lagrangian we scale $t \to Bt$ and $x \to B^\alpha x$. Find, based on our finding in the previous part, the exponent α (in terms of n) that guarantees that the new Lagrangian describes the same physical system.

(c) For the case of $n = 2$, show that your answer is consistent with the fact that the period of the motion is independent of the amplitude. (Note that, in this case, it may be better to consider the similarity transformation as: $x \to Dx$ and $t \to D^{1/\alpha}t$, with $D \stackrel{def}{=} B^\alpha$.)

(d) For the case of $n = 4$, we expect to have a bound oscillatory motion as well. In this case, will the period (τ) depend on the amplitude A? If so, how (i.e., what is β in $\tau \propto A^\beta$)?

(e) Now, consider a three dimensional problem with a potential that behaves as

$$U(\vec{r}) = a\, r^n$$

where $r = |\vec{r}|$. Use the same argument ($t \to Bt$, $\vec{r} \to B^\alpha \vec{r}$, $L \to CL$), and obtain the α exponent.

(f) For the Kepler problem, $n = -1$. Show that your answer is consistent with Kepler's third law (cf. page 303 of the textbook).

Problem 2 (15 points) **Two body problem.** We consider a two body problem with a *central force*, whose Lagrangian is given by

$$L = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

where two bodies with masses m_1 and m_2 interact ~~with~~ via a *central potential* $U(|\vec{r}_1 - \vec{r}_2|)$. A central potential is a potential that depends only on the relative *distance*.

(a) Use the following transformation to re-express the Lagrangian in terms of $\vec{R}$, $\dot{\vec{R}}$, $\vec{r}$, $\dot{\vec{r}}$, M and m.

$$\vec{R} \stackrel{def}{=} \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$\vec{r} \stackrel{def}{=} \vec{r}_1 - \vec{r}_2$$
$$M \stackrel{def}{=} m_1 + m_2$$
$$\frac{1}{m} \stackrel{def}{=} \frac{1}{m_1} + \frac{1}{m_2}$$

Here, $\vec{R}$ is the position vector of the center of mass and $\vec{r}$ is the relative position vector. M is the total mass, and m is the "reduced mass" (often called μ). Your answer should show explicitly that $\vec{R}$ and $\vec{r}$ are completely decoupled in the sense that

$$L \;=\; L_{cm}(\vec{R}, \dot{\vec{R}}) + L_i(\vec{r}, \dot{\vec{r}})$$

where L_{cm} is the part of the Lagrangian that depends on $\vec{R}$ and its time derivative, and L_i is the part of the total Lagrangian that depends on $\vec{r}$ and its time derivative. L_{cm} is the Lagrangian for the *center of mass* degrees of freedom $\vec{R}$, and L_i for the *relative (or internal)* degrees of freedom $\vec{r}$.

(b) Express the total momentum $\vec{P} \stackrel{def}{=} m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2$ in terms of the momenta associated with L_{cm} and L_i. Likewise, express the total angular momentum $\vec{L} \stackrel{def}{=} m_1\vec{r}_1\times\dot{\vec{r}}_1 + m_2\vec{r}_2\times\dot{\vec{r}}_2$ in terms of the angular momenta associated with L_{cm} and L_i. Discuss how your results make sense, using an example that consists of two particles interacting by a central potential, e.g. a binary star system.

(c) Which of these quantities are conserved and why? H (Hamiltonian), E (energy), $\vec{P}$ (total momentum), $\vec{P}_{cm}$ (center of mass momentum associated with L_{cm}), $\vec{P}_i$ (internal momentum associated with L_i), $\vec{L}$ (total angular momentum), $\vec{L}_{cm}$ (center of mass angular momentum), $\vec{L}_i$ (internal angular momentum).

(d) Consider a scenario that the potential is replaced by a more general form, $U(\vec{r}_1 - \vec{r}_2)$. How then do the answers to the previous part change? Discuss how likely this scenario is.

Problem 3 (20 points) A bead is constrained to move along a solid wire whose shape is given by

$$z \;=\; \frac{1}{2a}x^2 + \frac{1}{4a^3}x^4$$

where x and z are the coordinates of the xz plane where the wire resides. There is no friction. Here, a is a length scale, z is the vertical coordinate, and the surface gravity given by $-g\hat{z}$. The wire (and thus the xz plane) is being rotated at a constant angular frequency ω around the z axis.

(a) Find the Lagrangian, $L(x, \dot{x}, t)$.

(b) Find the canonical momentum, p_x, and the Hamiltonian H. Define the effective mass *function* $m_{eff}(x)$ such that $p_x = m_{eff}\dot{x}$. Is H conserved?

(c) Find the effective potential $U_{eff}(x)$ such that $H \equiv T_{eff} + U_{eff}$, where $T_{eff} = \frac{1}{2}m_{eff}\dot{x}^2$.

(d) Find all equilibrium points of the motion by solving $dU_{eff}/dx = 0$. Why does this method work even if m_{eff} is a *function* of x?

(e) Show that there is a critical frequency ω_c such that if $\omega < \omega_c$ then there is only one stable equilibrium point, while if $\omega > \omega_c$ then there are three equilibrium points of which two are stable and one is unstable. Find ω_c.

(f) Find the frequency for small oscillation around each stable equilibrium point for $\omega < \omega_c$ and $\omega > \omega_c$. Show that the frequency goes to zero, as $\omega \to \omega_c$ in each of these cases.

Problem 4 (20 points) **Rolling without slipping.** A slab of mass M is sliding without friction on a slope with angle α with respect to the horizontal. The slab is passing through a region of the slope, where there is a small opening through which the top of a wheel mounted in a cavity underground is exposed. The wheel is free to rotate. As the slab passes through this region, making contact with the wheel, the wheel rotates without slipping. The wheel's mass is m, its radius is R and its rotational inertia is given by $\gamma m R^2$, where γ is a number of order 1, determined by the shape (but not the size) of the wheel.

(a) Find the acceleration of the slab, as a function of g, α, M, m and γ. Find the value of the acceleration in the limits of $m \gg M$ and $M \gg m$ and show that your answer is reasonable in those limits.

(b) Is the energy conserved in this system? Explain.

(c) Find the force of constraint at the contact between the slab and the wheel. Identify the nature of this force. Calculate the power delivered to the slab by this force and show that it is not zero. Discuss your finding here in relation to the answer of the previous part, and explain how/why everything is fine.

Problem 5 (15 points) **Poisson bracket.** Within the Hamiltonian formalism, the Poisson bracket of two general functions $g(p_i, q_i, t)$ and $h(p_i, q_i, t)$ are defined as

$$[g, h] = \sum_i \left(\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i} \right)$$

Here, i is the index for generalized coordinates. Assuming that there are n generalized coordinates, g and h are functions of $2n + 1$ independent variables: $p_1, \cdots, p_n, q_1, \cdots, q_n, t$.

(a) Prove that $[g, h] = -[h, g]$, $[g, g] = 0$.

(b) Prove that $[q_i, p_j] = \delta_{ij}$ (Kronecker delta).

(c) Prove that $\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}$.

(d) Calculate $[\vec{L}_i, H]$ explicitly, where $\vec{L}_i$ and H are from problem 2 (parts a-c), and show that it is zero, consistent with what we just established in the previous part. [Hint: item 5 in page 2 of LN 4 may be useful.]

(e) Calculate $[\vec{P}_i, H]$ explicitly, where $\vec{P}_i$ and H are from problem 2 (parts a-c). Confirm that it is *not* zero, in general.

[Note: The Poisson bracket is to Classical Mechanics, what the "commutator" is to Quantum Mechanics. Their properties are very similar. In Quantum Mechanics, the definition of the commutator is $[A, B] = AB - BA$, where A, B are quantum mechanical "operators" (matrices, basically). H, q_i, p_i etc. become operators in Quantum Mechanics. All of the above properties have direct analogues in QM commutators (the only difference being the multiplicative constant $i\hbar$ appearing here and there in QM). In particular, the QM version of (b) is the canonical quantization condition, responsible for the Heisenberg uncertainty principle. The QM version of (c) is the Heisenberg equation of motion, completely equivalent to the Schrödinger equation of motion.]