Due Nov. 1, Tuesday.

Problem 1 (5 points) **A general property of the Lagrangian formalism.** Consider a Lagrangian L, and another Lagrangian L', which is related to the original one by

$$L'(q, \dot{q}, t) \ = \ L(q, \dot{q}, t) + f(q, \dot{q}, t)$$

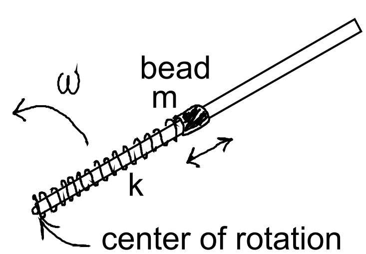
where

$$f(q, \dot{q}, t) = \frac{dF(q, t)}{dt}$$

Show that applying the Hamilton's principle $\delta S' = 0$ is completely equivalent to $\delta S = 0$. Here, S' is the action for L' and S for L. This means that the two equations of motions that one obtains through L' and L *must* be equivalent. Note carefully that, by definition, F should not be a function of $\dot{q}$, while f can be.

Problem 2 (10 points) **Simple pendulum in an accelerated frame.** Consider Example T7.6 (and Figure T7-4), which we went through in class. Now, suppose you are in the railroad car, and you are completely oblivious of what is going on outside the car. (If you like, you can imagine that the railroad car is really really big, and you simply never reached close enough to the boundary of the railroad car to know that there is an outside world.) So, it is not even possible for you to imagine that you are accelerating relative to something outside. However, you do know (by experience and experimentation) that the gravitational field is not vertical but given by $-a\hat{x} - g\hat{y}$, where $\hat{x}$ is the unit vector pointing to the right and $\hat{y}$ is the unit vector pointing up. (a) Find the Lagrangian within this reference frame inside the railroad car. Show that this Lagrangian and the Lagrangian that we found in class (LN 8, page 11) are different by a total time derivative of a function $F(q, t)$ as in problem 1 above. Find $F(q, t)$. ($q \stackrel{def}{=} \theta$ is recommended.) (b) Show that the equation of motion is completely unchanged from what we found in class (LN 8, page 11), as expected from problem 1.

Problem 3 (20 points) A bead is constrained to move, without friction, along a solid wire. The bead is connected to a spring with the spring constant k and the equilibrium length l. The spring wraps tightly around the wire, without any friction between the spring and the wire. The wire begins to rotate by a certain external force, with the other end of the spring fixed at the center of rotation, together with the end of the wire. The rotational motion of the wire reaches a constant angular velocity ω. We are interested in the motion *only after* this constant angular velocity is reached.



(a) Find the Lagrangian in terms of x, $\dot{x}$ and t, where x is the displacement of the spring relative to l.

(b) Find the canonical momentum, p, corresponding to x. What is the meaning of it?

(c) Find the Hamiltonian, H. Is it equal to the energy $E = T + U$?

(d) Is the Hamiltonian conserved? Explain your answer in one sentence. Is the energy conserved? Explain your answer.

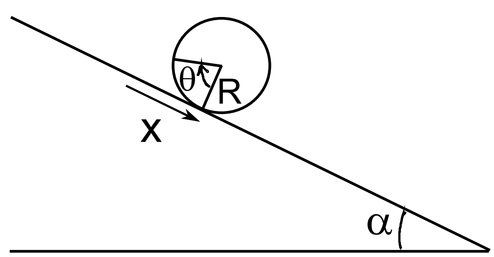
(e) By looking at H, identify the effective potential, U_{eff}, such that $H = \frac{1}{2}m\dot{x}^2 + U_{eff}(q)$.

(f) Now, consider ω as a parameter that can be varied from experiment to experiment. Show that there is a "critical" value of ω_c such that, a stable equilibrium exists only when $\omega < \omega_c$. Find ω_c, and find x_{eq}, the stable equilibrium point.

(g) Explain the Newtonian physics at the stable equilibrium point $x = x_{eq}$. Is there any other equilibrium point other than the stable equilibrium point?

(h) Find the angular frequency, ω_0, for small oscillation around the stable equilibrium point. How does the oscillation frequency behave as ω_c is approached? Keeping this and the equilibrium value of x in mind, sketch the potential energy as a function of x, when $\omega \ll \omega_c$ and $\omega \to \omega_c - 0^+$.

Problem 4 (15 points) An object with a circular crossection is rolling down on a slope, without slipping, as depicted here. In the figure below, x is the distance that the object travels on the slope, while θ is its angular displacement. The rotational inertia is given by $\gamma m R^2$, where m is the total mass and γ is an $O(1)$ constant, which depends on the details of the object ($\gamma = 1/2$ for a uniform solid cylinder, or 1 for a hollow cylinder, etc.) but not on m or R.



(a) Find the Lagrangian L. In this problem, you are required to express L as a function of $q, \dot{q}, t$, with *one* generalized coordinate q, which can be either x or θ. You must investigate and use the fact that $\theta = \theta(x)$ and $x = x(\theta)$.

(b) Find the canonical momentum, p, corresponding to the generalized coordinate that you defined. Discuss the meaning of the canonical momentum.

(c) Find the Hamiltonian, H. Is it equal to the energy $E = T + U$? Is it conserved?

(d) Find the equation of motion, and find the acceleration $\ddot{x}$ down the slope.

(e) Is there a friction force involved in this problem? Discuss the implication of your answer to your answer for (c).