

Due Oct. 20, Thursday.

Problem 1 (10 points) For each of the following statements, prove it if it is true, or provide one counter-example if it is not true. [*Note:* In this problem, we are considering functions of x and operators on those functions. If you like you can consider functions of t . I.e., you can change all x 's to t 's below, if you like. Changing x to t is more in line with Newton's equation (then $f(t)$ can be $x(t)$ or $v(t)$.)]

- (a) $Lf(x) = 3$. L is a linear operator.
- (b) if L is a linear operator, then L^n for any $n = 0, 1, 2, \dots$ is a linear operator. [Note that L^n defines a repeated application of L (if $n > 1$), or no application at all ($n = 0$).]
- (c) If L_1, L_2 are linear operators, then $\alpha L_1 - \beta L_2$ is also a linear operator, for any constants α, β . [Note: $(L_1 - L_2)f(x) \stackrel{def}{=} L_1f(x) - L_2f(x)$ and $(\alpha L)f(x) \stackrel{def}{=} \alpha g(x)$ where $g(x) = Lf(x)$.]
- (d) If L is a linear operator, then $\exp(L)$ is also a linear operator. Note: what is $\exp(L)$? It is $\sum_n \frac{L^n}{n!}$.
- (e) $Lf(x) = \ln f(x)$. L is a linear operator.
- (f) $Lf(x) = f(x)^3$. L is a linear operator.
- (g) $\cos\left(i\frac{d}{dx}\right)$ is a linear operator. Note: what is $\cos\left(i\frac{d}{dx}\right)$? It is $\sum_{n=even} \frac{1}{n!} \frac{d^n}{dx^n}$.

Problem 2 (10 points) Consider the following potential energy.

$$U(x) = a \left(\frac{b^{12}}{x^{12}} - 2 \frac{b^6}{x^6} \right)$$

where $a, b > 0$ are constants and $x > 0$.

- (a) Find all equilibrium points and discuss their stability.
- (b) Make a sketch of $U(x)$, noting the values at any equilibrium point and the limits of $x \rightarrow 0^+$ and $x \rightarrow \infty$.
- (c) What is the condition on the total mechanical energy E for a bound state motion to occur?
- (d) What is the period T for the simple harmonic motion near the minimum energy (assume mass m)?

Problem 3 (20 points) Consider the simple harmonic oscillator (SHO) problem

$$m\ddot{x} = -kx - 2m\beta\dot{x}$$

and its general solution for under-damping ($\omega_0 > \beta \geq 0$)

$$x = A \exp(-\beta t) \cos(\omega_1 t + \theta_0)$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

- (a) Find the expression for the kinetic energy T .
- (b) Find the expression for the potential energy U .
- (c) Find the expression for the total mechanical energy $E = T + U$ and make a sketch of $E(t)$ as a function of time, clearly noting the overall exponential decay time scale, the sinusoidal modulation period, and the overall sign of E .
- (d) Differentiate the above expression to obtain dE/dt . Verify that this is what you would expect from $dE/dt = dW_{nc}/dt$, where dW_{nc} is the infinitesimal work done by the non-conservative force of this problem.
- (e) **For the rest of this problem, assume $\beta \ll \omega_0$ and define $\langle Q \rangle$ as the time average of a quantity Q over one period $\tau = 2\pi/\omega_1$.** Find $\langle T \rangle$, $\langle U \rangle$, and $\langle E \rangle$. Note that the time average does not mean that $\langle Q \rangle$ is time independent. It is just coarse-grained.
- (f) Look up the “virial theorem” for power law forces, state it, and verify that relationship between $\langle T \rangle$ and $\langle U \rangle$ is in agreement with the theorem.
- (g) Show that the Q factor of this system can be written as

$$Q \approx 2\pi \frac{\langle E \rangle}{|\Delta E|}$$

where ΔE is the energy loss in one period τ .

Problem 4 (10 points) Consider a driven SHO with *no* damping.

$$\ddot{x} + \omega_0^2 x = A \cos(\omega t)$$

- (a) Find the general solution of this equation when $\omega \neq \omega_0$. Identify the complementary function $x_c(t)$ and the particular solution $x_p(t)$, and discuss the time dependence of each.
- (b) Find the general solution of this equation when $\omega = \omega_0$. Identify the complementary function $x_c(t)$ and the particular solution $x_p(t)$, and discuss the time dependence of each. Find the phase shift of $x_p(t)$. [Hint: assume $x(t) = g(t) \exp(i\omega t)$ for the particular solution of the complex-plane version of the above EOM and solve for $g(t)$.]

Problem 5 (10 points) Problem 3-26 of the textbook.