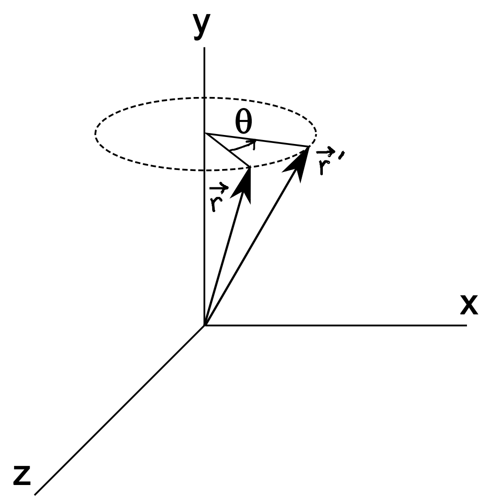
Due Oct. 6, Thursday.

Problem 1 (10pts) Consider a coordinate transformation by which a vector is rotated around the y axis by θ. Let us define the matrix $\overleftrightarrow{O}$ as

$$\vec{r}\,' = \overleftrightarrow{O}\,\vec{r}$$



(a) Find the matrix $\overleftrightarrow{O}$ as a function of θ. Do not assume that the matrix is an orthogonal matrix, prior to finding it, but do show briefly, after finding it, that it is an orthogonal matrix.

(b) Find all eigenvalues of $\overleftrightarrow{O}$. [Note: You need to find the determinant of a 3×3 matrix, and I trust that this topic was covered in your math/math-phys courses. The matrix at hand is "almost 2-dimensional" and so it should not be so bad.]

(c) (Extra Credit 5 pts; but you *must* read this one!) Prove that, in general, an eigenvalue, say λ, of an orthogonal matrix is a complex number of unit magnitude, $|\lambda| = 1$. [Make sure that your answer in part (b) agrees with this fact!] One important thing to keep in mind here is that the eigenvector of an orthogonal matrix is generally a complex vector. So, write down the eigenvalue equation for the matrix, and then consider its Hermitian conjugate, not its transpose, to start solving this problem. For a real matrix, such as an orthogonal matrix, its Hermitian conjugate is, of course, identical with its transpose. [Note: an orthogonal matrix is always diagonalizable, as it is a "normal matrix" – please look up what a normal matrix is, if you need to refresh your memory.]

Problem 2 (10 points) Find the perturbative solution to the following equation, (a) up to the first order in λ, (b) up to the second order in λ, and (c) up to the third order in λ, assuming $|\lambda| \ll 1$.

$$x - \frac{\pi}{3} = \lambda \cos x$$

Useful formulae: $\cos(a+b) = \cos a \cos b - \sin a \sin b$, $\cos\delta \approx 1 - \frac{\delta^2}{2} + O(\delta^4)$, $\sin\delta \approx \delta + O(\delta^3)$. where $|\delta| \ll 1$.

Problem 3 Consider tossing a ball upwards with initial speed v_0. There is air resistance $-mkv^2$ as well as gravity (the negative sign here means that the air resistance is opposite in direction to the velocity). While this problem is analytically solvable by brute force integration, here we won't do that. Instead, we will study this problem using the perturbation method only. The perturbation method is very useful for gaining insight, since it can be easily applied to any problem, whether it is exactly solvable analytically or not.

(a) (3 points) What is the dimension of k?

(b) (3 points) What is the terminal speed v_t? [Determine it using a free body diagram.] Check explicitly that the dimension of your answer is correct.

(c) (4 points) For small air resistance, what would be the dimensionless perturbation parameter that does not involve t? [This part may be done after (d).]

(d) (15 points) For small k, treat the air resistance as a small term in Newton's equation of motion (EOM), and calculate the leading order correction due to the air resistance for (i) the time that it takes for the ball to reach the maximum height, (ii) the maximum height, (iii) the time it takes for the ball to come back to the original vertical position, (iv) the speed when it comes back to the original vertical position. [Again, DO NOT solve the EOM exactly and then do a series expansion. Instead, use the perturbation method on the EOM. First, solve the EOM without the air resistance. Then, include the air resistance term, but plug in the first solution to the air resistance term of the EOM, to get the leading order correction to x.]

(e) (5 points) Make a sketch of your answer for part (iv) of (d), along with the exact answer (read prob 2-12 of the textbook). Compare the two curves and discuss.

(f) (Extra credit: 5 points) Use the work-energy theorem to re-derive, and thus verify, your answer for part (iv) of (d).

Problem 4 (10 points) A particle is released from rest and falls under the influence of gravity and air resistance. Find the relationship between v and the distance of falling y when the air resistance is equal to (a) αv and (b) βv^2. In both cases, show that your solutions reduce to the expected formula when the air resistance is turned off. The trick $a = dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$ may be useful for this problem.