

One letter-size crib sheet is allowed. No calculator. No other materials. Any questions to proctor are allowed.

$$\begin{aligned} \sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\ \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\ \sin(2a) &= 2 \sin a \cos a \\ \cos(2a) &= 2 \cos^2 a - 1 = 1 - 2 \sin^2 a \\ \sin \delta &= \delta + O(\delta^3) \\ \cos \delta &= 1 - \frac{1}{2} \delta^2 + O(\delta^4) \\ (1 + \delta)^\alpha &= 1 + \alpha \delta + \frac{1}{2} \alpha(\alpha - 1) \delta^2 + O(\delta^3) \\ \ln(1 + \delta) &= \delta - \frac{1}{2} \delta^2 + O(\delta^3) \\ e^\delta &= 1 + \delta + \frac{1}{2} \delta^2 + O(\delta^3) \end{aligned}$$

The last problem is an extra-credit problem, which will be counted towards supplementing your midterm score: it will be worth one problem of the midterm.

Show all your work. **Take time to read this exam.** Be neat in writing. Answers without explanations may get you very few points only, if any.

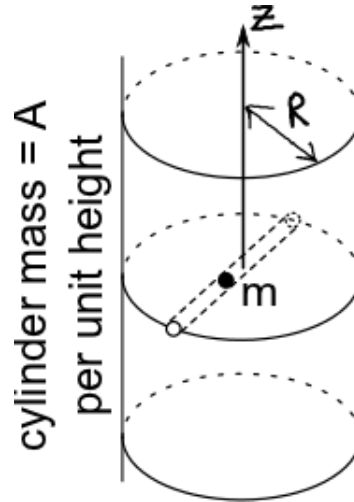
Believe that you are the one!! Good luck!

Problem 1 (60 points) A mass m is making a circle around a very heavy mass $M(\gg m)$, due to the gravitational force by M . There is another mass $m_2 = 2m$, which is also moving in the gravitational field of mass M , in the same plane of motion as m . It is observed that the two small masses (m and m_2) collide head-on. Just before the head-on collision, m has velocity \vec{v} , while $m_2 (= 2m)$ has velocity $-2\vec{v}$. Ignore any gravitational interaction between m and m_2 : this means that the two masses do not interact except during the collision. Assume that the collision is *elastic*.

- (a) What is the shape of the orbit for mass m_2 prior to the collision? Explain.
- (b) Find the velocities of m and m_2 right after the collision.

- (c) What are the shapes of the orbits for mass m_2 and m after the collision? Explain.

Problem 2 (80 points) An infinitely long cylinder has mass A per unit length (or height; see below).

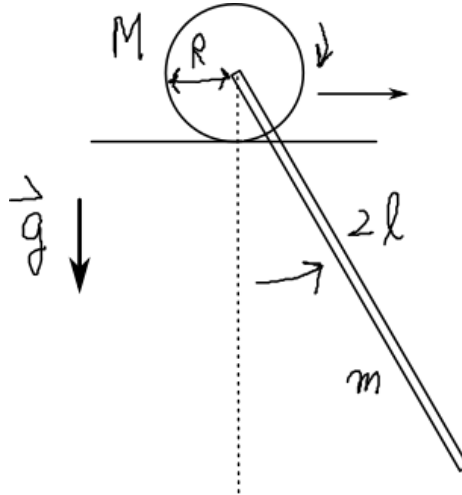


This problem is cumulative: assumptions made in one part apply to parts later to it, but not to parts prior to it. Note also that parts (d-f) can be done without having to provide solutions to parts (a-c), simply by assuming a generic form of potential that depends on the statement (i) of part (c).

- Find the gravitational field \vec{g} due to this cylinder, inside and outside the cylinder.
- Find the gravitational potential Φ corresponding to \vec{g} , inside and outside the cylinder.
- A small hole is drilled as pictured above, perpendicular to the z axis of the cylinder. Assume that there is no friction between the hole and the mass m . (i) Show that the motion of a test mass m inside the hole is a simple harmonic motion. (ii) Find the time it would take for the mass m to travel all the way to the other side of the hole when it is dropped with zero speed into the hole.
- Additionally, the system is rotating at an angular speed ω_0 around the z axis. Find the effective potential of this problem, while assuming that mass m is always inside the hole.
- The origin – i.e., the point where the z axis intersects the hole path – is an equilibrium point of this system. Is there a critical angular speed above which it becomes an unstable equilibrium? If so, find it. If not, prove that the origin is always a stable equilibrium point.

- (f) Find the normal force that the surface of the hole exerts on mass m as a function of the position and the velocity.

Problem 3 (60 points) A hoop of mass M and radius R rolls without slipping on a rail. Hanging freely from its center (through some massless axles and other massless auxiliary parts that we are not interested in the details of) is a uniform thin rod of length $2l$ and mass m (so its moment of inertia is $ml^2/3$). The rod can swing freely as shown. Note that the pivot point of this swinging motion is the center of the hoop, and is moving in general.



Assume small oscillations.

- (a) Find normal modes (frequency, \vec{T} and diagram).
- (b) Assume that the system is found to be in equilibrium. Then, you raise the rod to a small angle θ_0 , with everything else held in place. And you quickly take your hands off of everything. Find the complete mathematical description of the system as a function of time.

Problem 4 (Extra credit; 50 points) Find the normal modes of the following coupled oscillator system, assuming small oscillations.

Here the system consists of four identical beads of mass m that can slide without friction on two stiff and smooth perpendicular wires, which lie fixed in space. There is no gravity in this problem. Each neighboring pair of beads are connected by a spring of spring constant k . In addition, two diagonal springs with spring constant κ connect the two diagonal pairs of beads, as shown. [These diagonal springs are mounted in a way (e.g. one below the wire and the other above the wire) so that they do not touch each other. They also exert no friction on wires.] In equilibrium, the four beads form a square shape. For each normal mode, find the normal mode frequency, the \vec{T} vector, and the diagram of the normal mode

motion. As always, in particular for this one, answers without explanations may get you very few points, if any.

