Review-①

Overarching rule --- No contact, no force (except gravity, magnetic force, ...) (~~static, van-der Waals, ...~~)

Lagrangian mechanics

- $L = T - U$
- T in polar coord. spherical coord.
- $-\dfrac{\partial L}{\partial q} + \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} = 0$ ----- (✱)
- Constraint(s) $\quad a_t + \sum_i a_i \dot{q}_i = 0$

$$-\frac{\partial L}{\partial q_i} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \underbrace{a_i \lambda}_{\text{force of constraint}} \quad \text{or} \quad \underbrace{\sum_s}_{\text{if multiple constraints}} a_{s,i}\lambda_s$$

- Conservation principles (no constraints)

$H = \sum_i p_i \dot{q}_i - L$ conserved if no explicit t dependence in L (or H)

$L = L(q_i, \dot{q}_i, t) \xrightarrow{i} q_i\ \dot{q}_i$: collective index

$\underbrace{L \neq L(q_i)}_{\text{no explicit dependence on } q_i}$ ($i \rightarrow$ particular i) $\Rightarrow p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ is conserved. [from (✱)]

- Closed system ... $\underbrace{\vec{P}}_{\text{total}}, \underbrace{\vec{L}}_{\text{total}}, \underbrace{H = E}_{\text{total}}$ conserved.

$\vec{P} = M\dot{\vec{R}}$, $\quad \vec{L} = \vec{L}_M + \vec{L}_{int,cm}$, $\quad E = T + U$, $\quad T = T_M + T_{int,cm}$

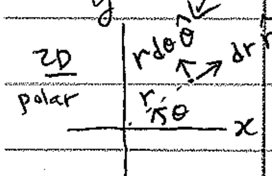
- U_{eff}? $\quad H$ (or E) when conserved

$$= \underbrace{K}_{\text{kinetic energy (for one } \dot{q})} + \underbrace{U_{eff}(q)}_{U + \text{centrifugal term etc.}}$$

Why a big deal? $\rightarrow K = A\dot{q}^2$, $U_{eff} = U_{eff}(q)$

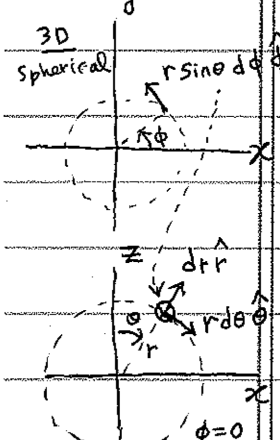
$\left(A = \frac{1}{2}m^{*}\right)$ $\quad \rightarrow$ can solve for $q(t)$ or rather $t(q)$

2D polar



$d\vec{r} = dr\,\hat{r} + rd\theta\,\hat{\theta}$

3D spherical



$d\vec{r} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$

- $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$ "Jacobian" $= r^2\sin\theta$
- $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$
- $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$

Central force problem

- $U_{eff}(r) = ?$ When does a circular motion occur?
- Angular momentum conservation --- Kepler's 2nd
- Kepler problem $U(r) = -\frac{k}{r}$

$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta$ $\tau^2 \propto a^3$ for elliptical orbits

1st law $0 \le \varepsilon < 1$ 3rd law a : semi-major axis

- Some general properties that can be "derived" or recalled from the circular motion?
  - centrepetal force equation $\mu \frac{v^2}{R} = \frac{k}{R^2}$
  - E-conservation $\frac{1}{2}\mu v^2 - \frac{k}{R} = E$

$\rightarrow \tau^2 = \cdots a^3$

$\rightarrow 2\langle T \rangle = -\langle U \rangle$

$\rightarrow \langle E \rangle = -\frac{k}{2a}$

valid for elliptical orbits

Coupled oscillators

- $T = \frac{1}{2} \dot{\vec{q}}^{\,t} \overleftrightarrow{M} \dot{\vec{q}}$ $\overleftrightarrow{M}, \overleftrightarrow{A}$: symmetric

$U = \frac{1}{2} \vec{q}^{\,t} \overleftrightarrow{A} \vec{q}$

$(\overleftrightarrow{A} - \omega^2 \overleftrightarrow{M}) \vec{T} = 0 \rightarrow \vec{q} = \vec{T}_i \eta_i$

eigenvalue ω_i eigenvector $\vec{T}_i$ for the i-th normal mode

as many i's as q_i's $\eta_i = D_i \cos(\omega_i t + \phi_i)$

$\rightarrow$ diagonalize $\rightarrow$ η_i's (normal modes)

$$\vec{q} = \overleftrightarrow{T} \vec{\eta} = \begin{bmatrix} \vec{T}_1 \cdots \vec{T}_n \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \qquad \vec{\eta} = \overleftrightarrow{T}^{-1} \vec{q}$$

$\overleftrightarrow{T}$? usually (almost) orthogonal.

- other (more general) methods of finding the normal modes

① Assume $q_j = u_j e^{i\omega_i t}$ (or $e^{i\omega t}$) → solve for u_j's — equivalent to calculating $\vec{T}_i$ or $\vec{T}$

② By "intuition" or "experience" or "inspection of symmetry," one can write down solutions and then prove them.
--- Proving amounts to showing that each q_j oscillates with the same ω and obtaining ω. "Writing down solutions" amounts to writing down $\vec{T}$'s.

①, ② are more generally applicable even to problems for which the "$\overleftrightarrow{M}$, $\vec{A}$ method" is not applicable (e.g. when there is a damping).

For general problems, esp. rigid body problems

EoM set 1 $\left(\begin{array}{l} \dot{\vec{P}} = \vec{F}_{ext} \\ \dot{\vec{L}}_{int,cm} = \vec{N}_{ext} \end{array} \right.$

~~These~~ are valid even if the c.m. frame is generally not an inertial frame.

EoM set 2 $\left(\begin{array}{l} \dot{\mathbf{P}}_M = \vec{F}_{ext} \\ \dot{\vec{L}}_{int,cm} = \vec{N}_{ext} \end{array} \right.$

EoM set 3 $\underbrace{\dot{\vec{L}}}_{\text{total}} = \vec{N}_{ext}$ in an inertial frame

τ torque.

Rigid body

$$\overleftrightarrow{I} = \sum_i m_i \begin{bmatrix} y_i^2+z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & z_i^2+x_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2+y_i^2 \end{bmatrix}$$

or $$\int dm \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

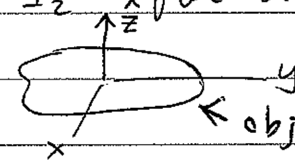
Principal axes of inertia ... mutally orthogonal axes that diagonalizes

$$\overleftrightarrow{I} \dashrightarrow \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$\uparrow$

We work with this as a rule.
Need to know how to identify them and then to calculate I_1, I_2, I_3.

Helpful) ① $I_3 = I_1 + I_2$ for a planar object.



← object in the xy plane

② $\overleftrightarrow{I} = \underbrace{\overleftrightarrow{I}_M}_{} + \underbrace{\overleftrightarrow{I}_{int,cm}}_{}$

(Parallel axis theorem)

$\overleftrightarrow{I}_{int,cm}$: the "usual" $\overleftrightarrow{I}$ in the cm frame

$\overleftrightarrow{I}_M$: $M \begin{bmatrix} y_{cm}^2+z_{cm}^2 & -x_{cm}y_{cm} & -x_{cm}z_{cm} \\ -x_{cm}y_{cm} & z_{cm}^2+x_{cm}^2 & -y_{cm}z_{cm} \\ -x_{cm}z_{cm} & -y_{cm}z_{cm} & x_{cm}^2+y_{cm}^2 \end{bmatrix}$

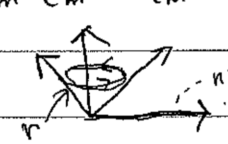
For rotations

$\vec{L} = \overleftrightarrow{I}\vec{\omega}$

$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}^{\,t}\,\overleftrightarrow{I}\,\vec{\omega}$

May Need to add T_M or $\vec{L}_M$ to get the total.

- $\left(\dfrac{d\vec{A}}{dt}\right)_{nr} = \left(\dfrac{d\vec{A}}{dt}\right)_r + \vec{\omega}\times\vec{A}$



- Rolling w/o slipping : $\vec{v}$ at contact point $= 0$ relative to the surface on which the thing rolls

Gravity (Newton's laws and Gauss laws)

$$\int d\vec{S}\cdot\vec{g} = -4\pi G M_{inside}$$

Review - 5

Collisions

$$-\vec{\nabla}\Phi = \vec{g}\ , \quad \vec{F} = m\vec{g}$$

- Momentum conserved.
- Angular momentum conserved.

) for a closed system

for open system too, if external force is not large ($\because$ collision time is short)

- Kinetic energy is generally not conserved. (inelastic collision)

Unless it is said to be an elastic collision (or it is completely obvious), assume inelastic.

- Coefficient of restitution $\varepsilon = \dfrac{|\vec{v}_1 - \vec{v}_2|}{|\vec{u}_1 - \vec{u}_2|}$ for head-on collision.

  u: initial
  v: final

- $T'(= T_{int,cm}) = \frac{1}{2}\mu\left|\dot{\vec{r}}_1 - \dot{\vec{r}}_2\right|^2$ is cool to remember.

- On collision, a violent exchange of 3rd law forces impart impulse to a particle

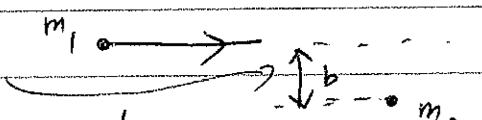
$$\vec{I} = \vec{F}_{ave}\cdot\Delta t = \Delta\vec{p} \text{ for that particle}$$

The torque delivered to that particle

$$\vec{r}\times\vec{I}/\Delta t$$

$$\therefore\ \Delta\vec{L} = \vec{r}\times\vec{I}$$

- Impact parameter (b)



$L' = \mu u' b$

related to angular momentum

reason for non-head-on collisions.