1. A particle in a central potential, $V(r)$, in three dimensions, can be described in terms of a one-dimensional problem with an effective potential, $U_{eff}(r)$,

$$U_{eff}(r) = V(r) + \frac{L^2}{2mr^2}$$

where L is the angular momentum.

a. Derive this expression. (Hint: you may want to keep in mind that motion in a central potential is in a plane).

b. For a *circular* orbit, determine r for a potential $V(r) = Cr^x$, for a given value of L. (If $x > 0$, $c > 0$; if $x < 0$, $c < 0$).

c. What is the condition on x for the stability of the orbit?

A particle in a central field moves in the spiral orbit $r = c\theta^2$ where c is a constant. For initial angular momentum L and mass m, determine:

a. the form of the force law.
b. how the angle θ varies with time t.

Consider two point masses m_1 and m_2 connected by a massless string. m_1 can move in a horizontal plane on a frictionless table. The string connecting the two goes through a frictionless hole in the table, and m_2 can only move in the vertical direction. The length of the string equals the height of the table. m_1 has been set in motion about the hole in such a way that the masses never reach either the floor or the hole.

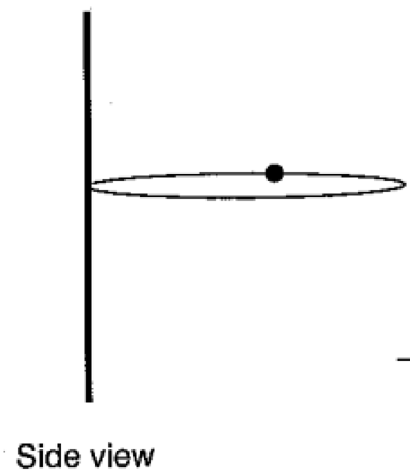
a) Write a Lagrangian for the system. Include any constraints explicitly.

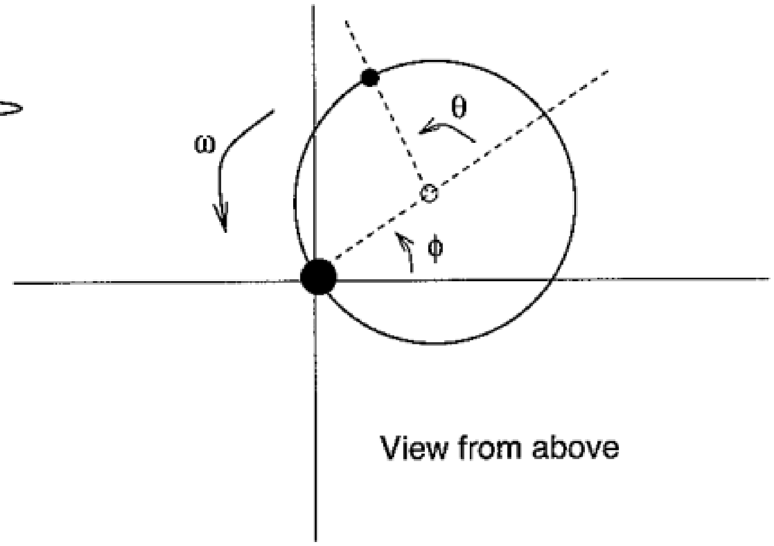
b) Write the equations of motion.

c) Find two integrals (constants) of motion. What do they represent physically?

d) Find an equilibrium point for m_2 and the frequency of small oscillations about this equilibrium point.

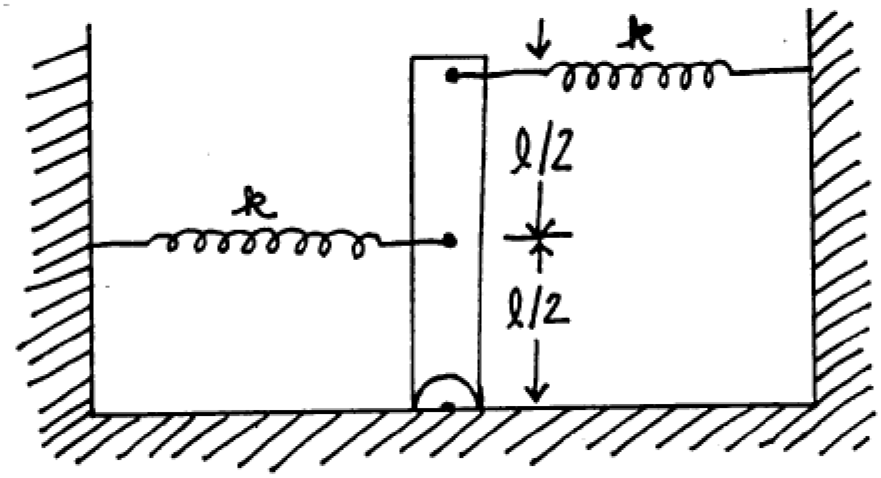
A bead of mass m slides frictionlessly on a wire loop of radius R. The loop is attached at one point to a vertical rotating rod ($\phi = \omega t$) so that the loop rotates in the horizontal plane about a point on its rim.





(A) Derive the equation of motion for the angle θ that the bead makes relative to the diameter pointing toward the rod (see figure).

(B) Find an equation for the radial force constraining the bead to on be the loop, as a function of θ, $\dot{\theta}$, and the constants m, R, and ω.



A rod of length l and mass M pivots at one end. It is held by two springs one at the top and the other at the mid height. Both springs have spring constant k and pull perpendicular to the rod at vertical equilibrium. What is the angular frequency for small oscillations about the equilibrium position? Gravity can be neglected.

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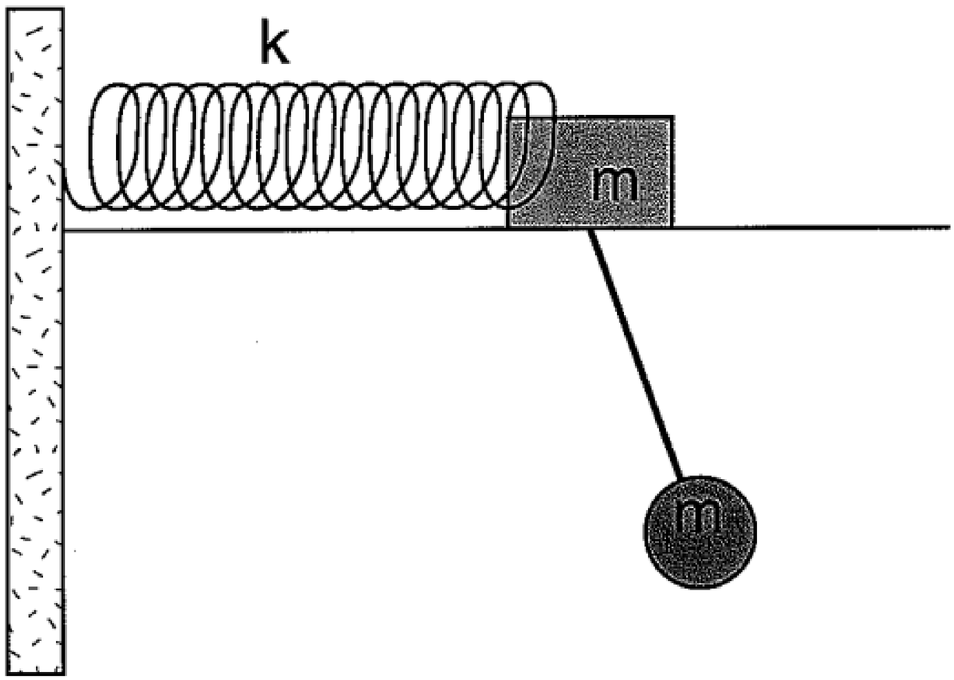
A block of mass m is connected to a massless spring of spring constant k such that it can oscillate horizontally. From the block is suspended a simple plane pendulum of length ℓ and mass m. Assume that all of the motion takes place in a single plane with no friction.



(A) Choose appropriate generalized coordinates and express the kinetic energy T in terms of them.

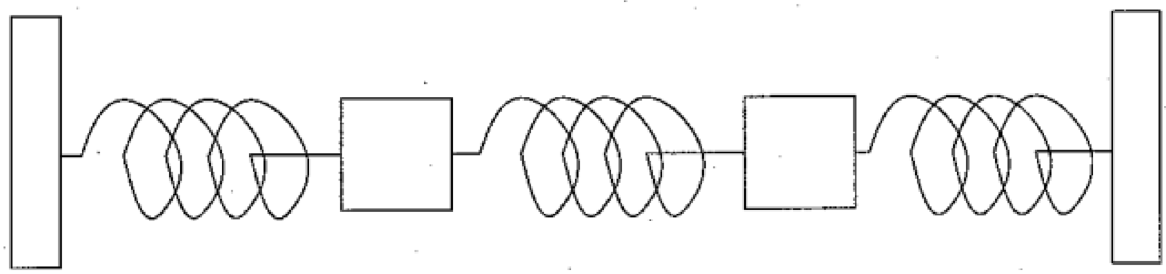
(B) Express the potential energy U in terms of the generalized coordinates.

(C) Derive the equations of motion from Lagrange's equations.

(D) Expand the kinetic and potential energies about the equilibrium point to arrive at the constant tensors m and A such that $T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k$ and $U = \frac{1}{2}\sum_{j,k} A_{jk}q_j q_k$.

(E) Solve for the two eigenfrequencies of small oscillations. Do not find the corresponding eigenvectors.

Consider the linear system of masses and springs shown below. The spring constants for the two outer springs are g, while the inner spring has spring constant h. The masses of the blocks are equal (m).



(A) Suppose that both masses start from their equilibrium positions. If the left-hand mass has an initial velocity v_0 toward the right-hand mass, calculate the position of the right-hand mass as a function of time. (At some point in your solution, describe clearly and succinctly the basic approach and procedure you are following.)

(B) Suppose that the left-hand mass is displaced a distance x_0 away from its equilibrium position toward the right-hand mass. Calculate the position of the right-hand mass as a function of time.

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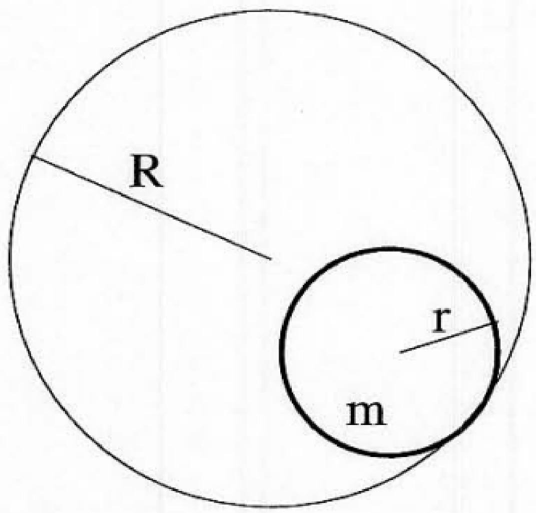
A hollow cylinder of circular cross section, radius r and mass m rolls, under the force of gravity, without friction and without slipping inside a fixed cylinder of circular cross section and radius R.

a. Write the Lagrangian for this system.

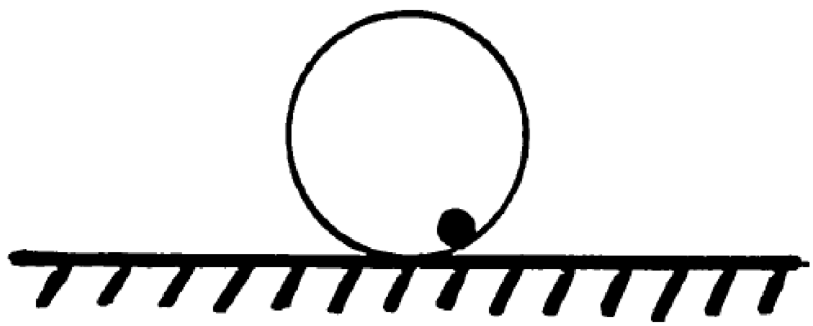
b. Find the equation of motion.

c. Find the frequency of small oscillations.

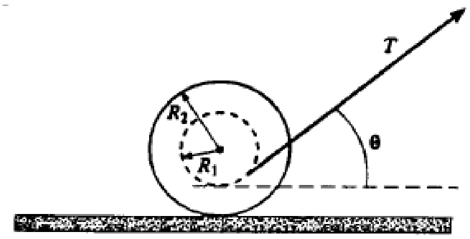
d. Let us say that we require for the small cylinder to reach the top without falling. Find the minimum speed of the center of mass of the small cylinder when it is at the bottom, to meet this requirement.



A point mass m is attached to the rim of an otherwise uniform hoop of mass M and radius R. Calculate the frequency of small-amplitude rolling oscillations.



A spool rests on a rough table as shown. A thread wound on the spool is pulled with force T at angle θ.



a) Show that there is an angle θ for which the spool remains at rest.

b) At this critical angle find the maximum T for equilibrium to be maintained. Assume a coefficient of friction μ.

A billiard ball is a spherical object with a uniform density.
(a) Calculate the inertia tensor for it.
(b) You hit it with a cue. Assume that the cue is moved along the horizontal direction only, and so the applied force is horizontal. Where should you be hitting in order to minimize the energy loss of the ball as it rolls without slipping or slides while rolling? Assume that you can hit anyhwere, not just on the vertical line of symmetry. [Hints: The minimum energy loss occurs for rolling without slipping. The answer is not a point, but a line.]