

PH102, 2013W, Lecture Notes: January 24, Thurs, Class 6

Angular Momentum in External Magnetic Fields

Objectives:

- Differentiate how an electron's total angular momentum is constructed in the presence of strong and weak magnetic fields.

An electron revolving around a nucleus in an atom has two types of angular momentum:

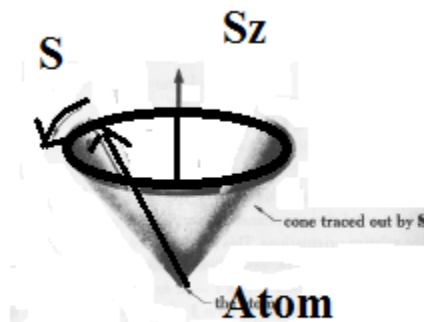
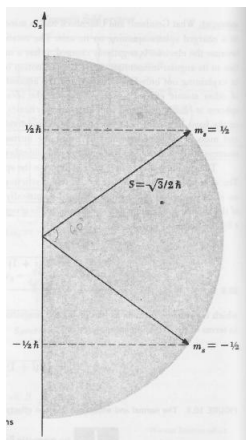
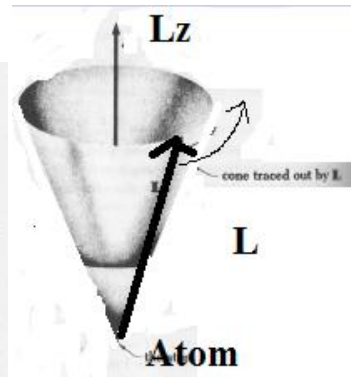
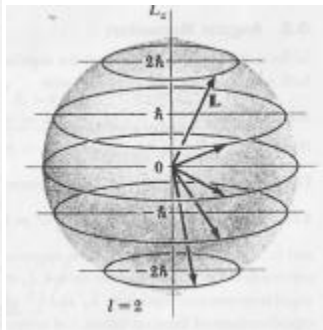
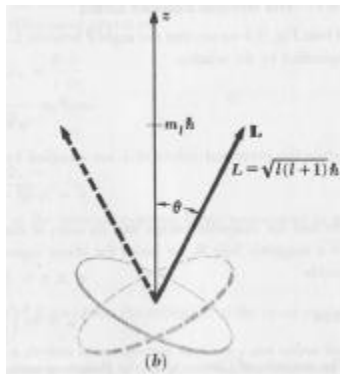
- Orbital angular momentum: \vec{L}
- Spin angular momentum: \vec{S}

$$|\vec{L}| = \sqrt{l(l+1)}\hbar \quad \text{where } l = 0, 1, 2, \dots, n-1$$

$$L_z = m_l \hbar \quad \text{where } m_l = -l, -l+1, \dots, l-1, l$$

$$|\vec{S}| = \sqrt{s(s+1)}\hbar \quad \text{where } s \text{ is a number intrinsic to a given particle}$$

$$S_z = m_s \hbar \quad \text{where } m_s = -s, -s+1, \dots, s-1, s$$



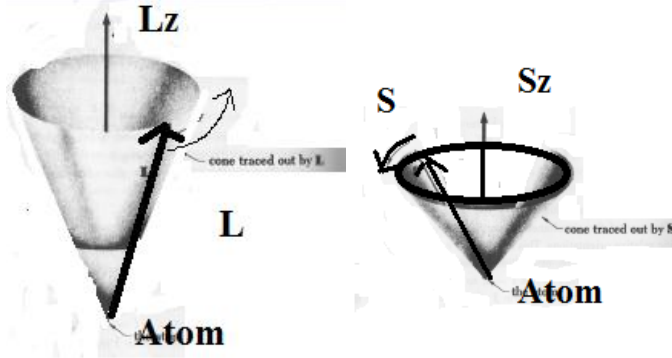
In the presence of an external magnetic field, \vec{B} , we need to consider

- Magnetic dipole moment related to orbital angular momentum of an electron, $\vec{\mu}_L = -\left(\frac{e}{2m}\right)\vec{L}$

- Magnetic dipole moment related to spin angular momentum of an electron, $\vec{\mu}_S = -\left(\frac{e}{m}\right)\vec{S}$

The Paschen-Back Effect:

When the external magnetic field \vec{B} is strong, then the \vec{L} and \vec{S} are separately aligned with the external magnetic field.



Energy related to the magnetic moment due to \vec{L}

$$U_L = -\vec{\mu}_L \cdot \vec{B} = -\mu_{L_z} B_z = -\left(-\frac{e}{2m} L_z\right) B_z = \frac{e}{2m} m_l \hbar B_z$$

Energy related to the magnetic moment due to \vec{S}

$$U_S = -\vec{\mu}_S \cdot \vec{B} = -\mu_{S_z} B_z = -\left(-\frac{e}{m} S_z\right) B_z = \frac{e}{m} m_s \hbar B_z$$

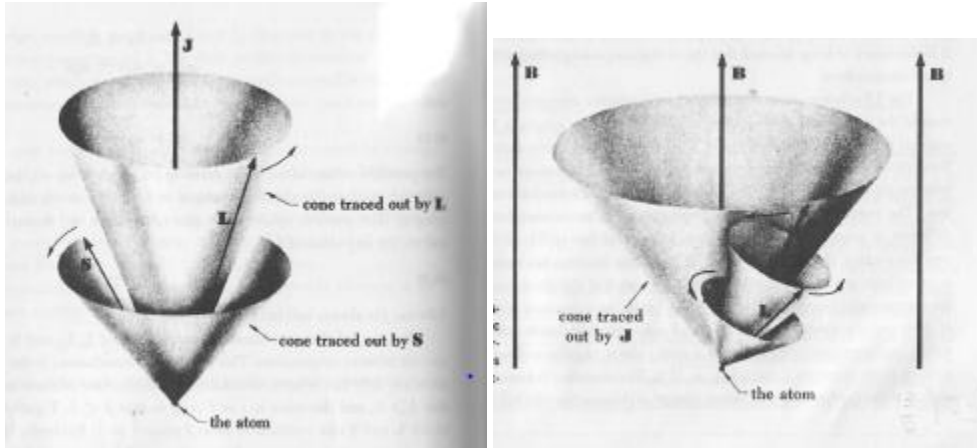
Therefore E_n will split into:

$$E_{\text{strong magnetic field}} = E_n + \frac{e}{2m} (m_l + 2m_s) \hbar B_z$$

In this case, n, l, m_l, m_s are good quantum numbers.

The Zeeman Effect:

When the external magnetic field \vec{B} is weak, then the \vec{L} and \vec{S} of an electron are coupled into a total angular momentum, $\vec{J} = \vec{L} + \vec{S}$



Therefore, the total magnetic moment can be written as

$$\vec{\mu}_J = \vec{\mu}_L + \vec{\mu}_S = -\left(\frac{e}{2m}\right)(\vec{L} + 2\vec{S})$$

Note that $\vec{J} = \vec{L} + \vec{S}$ and $\vec{\mu}_J = -\left(\frac{e}{2m}\right)(\vec{L} + 2\vec{S})$ are not aligned. We need to project the $\vec{\mu}_J$ onto the \vec{J} . Consider the angle between $\vec{\mu}_J$ and \vec{J} as $\pi + \delta$.

Then, $|\vec{\mu}_J||\vec{J}|\cos(\pi + \delta) = \vec{\mu}_J \cdot \vec{J}$ and $\cos \delta = -\frac{\vec{\mu}_J \cdot \vec{J}}{|\vec{\mu}_J||\vec{J}|}$

$$\begin{aligned} |\vec{\mu}_J| &= |\vec{\mu}_J| \cos \delta = |\vec{\mu}_J| \left(-\frac{\vec{\mu}_J \cdot \vec{J}}{|\vec{\mu}_J||\vec{J}|} \right) = -\frac{\vec{\mu}_J \cdot \vec{J}}{|\vec{J}|} = \frac{e}{2m} \frac{(\vec{L} + 2\vec{S}) \cdot (\vec{L} + \vec{S})}{|\vec{J}|} \\ &= \frac{e}{2m} \frac{|\vec{L}|^2 + 2|\vec{S}|^2 + 3\vec{L} \cdot \vec{S}}{|\vec{J}|} = \end{aligned}$$

$$\text{since } \vec{L} \cdot \vec{S} = \frac{1}{2}(|\vec{J}|^2 - |\vec{L}|^2 - |\vec{S}|^2)$$

$$|\vec{\mu}_J| = \frac{e}{2m} \frac{3|\vec{J}|^2 - |\vec{L}|^2 + |\vec{S}|^2}{2|\vec{J}|} = \frac{e}{2m} \frac{3j(j+1) - l(l+1) + s(s+1)}{2\sqrt{j(j+1)}} \hbar$$

Consider $\vec{B} = B_z \hat{z}$, then potential energy and since $\cos \theta = \frac{J_z}{|\vec{J}|} = \frac{J_z}{\sqrt{j(j+1)}\hbar}$

$$\begin{aligned} U = -\vec{\mu}_J \cdot \vec{B} &= |\vec{\mu}_J| B_z \cos \theta = |\vec{\mu}_J| B_z \frac{J_z}{\sqrt{j(j+1)}\hbar} = \frac{eJ_z B_z}{2m} \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} = \\ &= g_{Lande} \frac{eJ_z B_z}{2m} = g_{Lande} \frac{eB_z}{2m} (m_j \hbar) \end{aligned}$$

$$\text{Where } g_{Lande} = \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$$

Note that

- if $s=0$ and $j=l$, $g_{Lande} = 1$
- If $l=0$ and $j=s$, $g_{Lande} = 2$
- Therefore, the degenerate E_n of the hydrogen atom is modified under a weak magnetic field as
- $E_{weak\ external\ magnetic\ field} = E_n + g_{Lande} \frac{eB_z}{2m} (m_j \hbar)$
- In this case, n, l, j, m_j are good quantum numbers

Then, what counts as a strong and weak external magnetic field? We use the interaction (L and S) between the magnetic field generated by the orbiting electron and the electron's spin.

$$U = -\vec{\mu}_S \cdot \vec{B}_{due\ to\ \vec{L}}$$

$$\vec{B} = \frac{\mu_0 e}{4\pi m r^3} \vec{L}$$

For $2p$ state, $l = 1$, $n = 2$, and $r = n^2 a_0$

$$B = \frac{\mu_0 e}{4\pi m (4a_0)^3} \sqrt{1} \cdot 2\hbar = 0.28\text{ T}$$

Associated energy level $U = \mu_S B = 2 \times 10^{-5}\text{ eV}$

Therefore, the energy associated with LS coupling is very small.

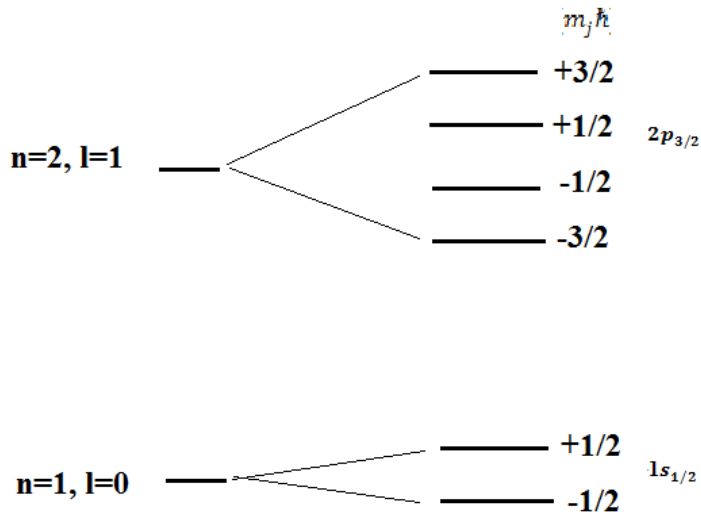
A magnetic field that can interrupt the LS coupling, therefore, should be much larger than .28T.

Example : $2p_{3/2} \rightarrow 1s_{1/2}$ in a weak external magnetic field of .05T

For $1s_{1/2}$, $l = 0$, $s = \frac{1}{2}$, $j = \frac{1}{2}$; $m_j = -\frac{1}{2}, \frac{1}{2}$

For $2p_{3/2}$, $l = 1$, $s = \frac{1}{2}$, $j = \frac{3}{2}$; $m_j = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$

The split due to LS coupling is shown in the diagram:



We can calculate the amount of deviation occurs from the E_n value in each orbital by first calculating g_{Lande}

$$g_{Lande} = \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$$

For $1s_{1/2}$, $l = 0, s = \frac{1}{2}, j = \frac{1}{2}$; $g_{Lande} = 2$;

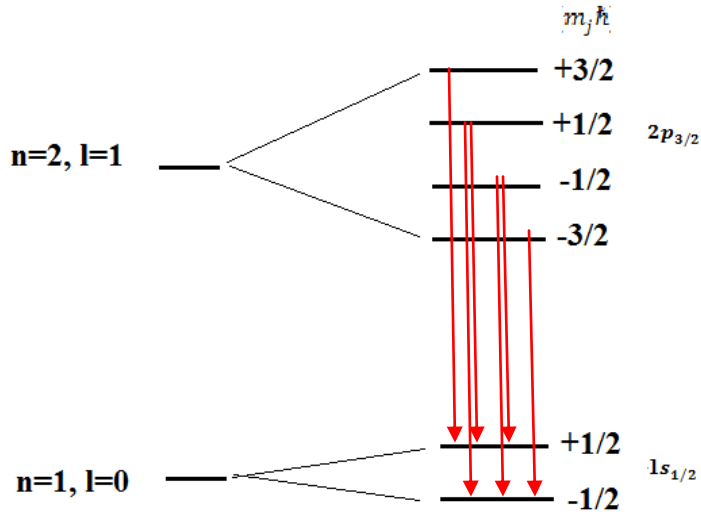
$$\Delta E = g_{Lande} \frac{eB_z}{2m} (m_j \hbar) = \frac{2eB_z \hbar}{2m} (m_j) = \frac{2eB_z \hbar}{2m} \left(\pm \frac{1}{2} \right)$$

For $2p_{3/2}$, $l = 1, s = \frac{1}{2}, j = \frac{3}{2}$; $g_{Lande} = \frac{4}{3}$

$$\Delta E = g_{Lande} \frac{eB_z}{2m} (m_j \hbar) = \left(\frac{4}{3} \right) \frac{eB_z \hbar}{2m} (m_j) = \frac{2eB_z \hbar}{3m} \left(\pm \frac{3}{2}, \pm \frac{1}{2} \right)$$

Transition between the two states can occur when $\Delta m_j = 0, \pm 1$ (because photon's spin is 1 and the total angular momentum should be conserved before and after the transition).

This means that only the following transitions are allowed:



Therefore, a photon can have the energy difference:

$$\Delta E = \Delta E_{\text{without magnetic field}} + \frac{2eB_z \hbar}{2m} \left(\pm \frac{5}{3}, \pm \frac{3}{3}, \pm \frac{1}{3} \right) \rightarrow \text{therefore 1 spectral line splits into 6 spectral lines in the presence of a weak magnetic field. The energy order of the split is } 1.9 \times 10^{-6} \text{ eV. Compare this with the difference between } E_2 - E_1 = -13.6 \left(\frac{1}{4} - 1 \right) \text{ eV} = 10.2 \text{ eV.}$$