PH102: Interactive Lecture 3

- Topics
  - 3d Schrodinger Equation for Hydrogen atom
  - $(x, y, z) \leftrightarrow (r, \theta, \phi)$
  - Separation of variables $R\Theta\Phi$
  - Three equations
  - Three quantum numbers
  - Wave functions
  - Quantization of angular momentum (L)
  - Degeneracies
  - Normalization
  - Angular whereabouts

Quantum Problem Solving Schema

- Think about potential energy and Hamiltonian
- Divide regions, if possible
- Choose appropriate coordinates
- Write Schrodinger Equation for each region
- Separation of variables, if possible
- Set “constant”s if applicable
- Solve for wave function
  - Boundary conditions
  - Normalization
- Solve for energy
- Build energy level diagrams and inspect for energy degeneracies

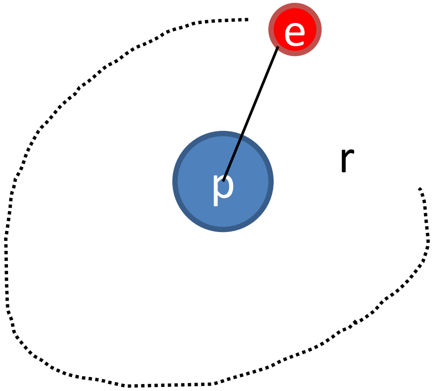
Hydrogen atom

- Potential created by Coulomb interactions between electron ($-e$) and proton ($+e$)

Hydrogen atom

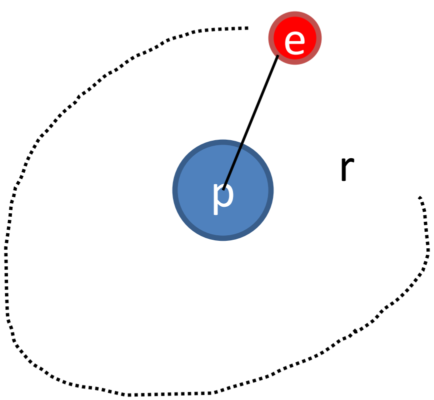
- Potential created by Coulomb interactions between electron (-*e*) and proton (+*e*)

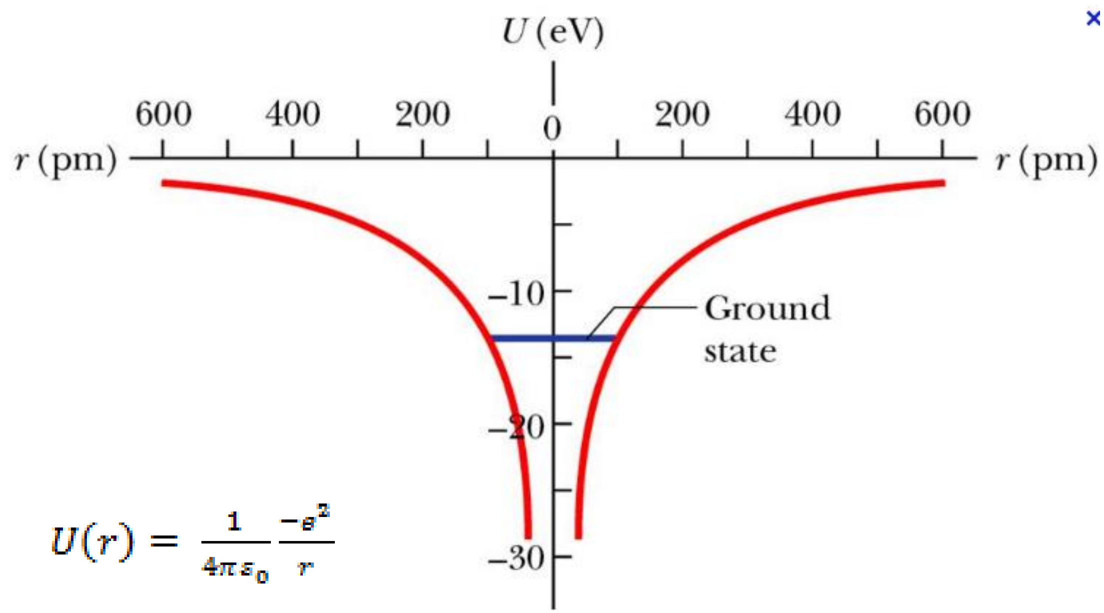
$$U(r) = \frac{1}{4\pi\varepsilon_0}\frac{-e^2}{r}$$



Hydrogen atom

- Potential created by Coulomb interactions between electron (-*e*) and proton (+*e*)



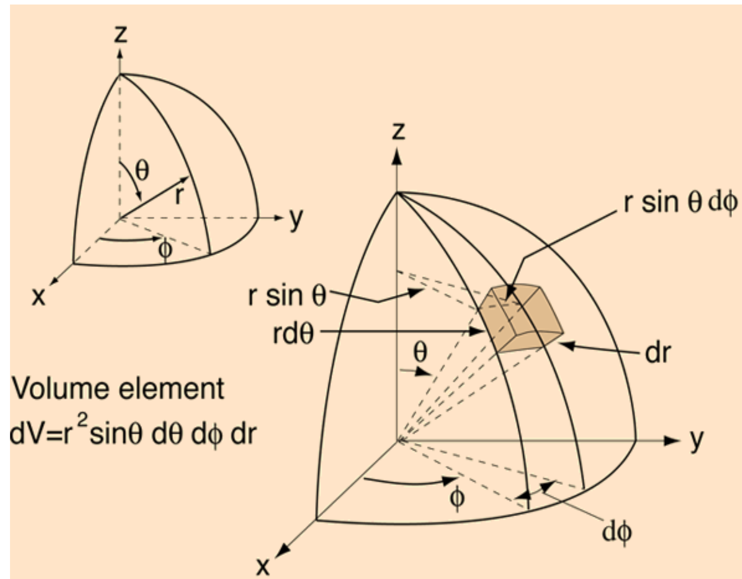


$$U(r) = \frac{1}{4\pi\varepsilon_0}\frac{-e^2}{r}$$

$$(x, y, z) \quad \leftrightarrow \quad (r, \theta, \phi)$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ \theta = \cos^{-1}\dfrac{z}{r} \end{cases}$$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$



$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \csc\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \csc^2\theta\,\frac{\partial}{\partial\phi^2}\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi^2}$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

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Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

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$$\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = -\frac{2mr^2}{\hbar^2}(E-U)\,\psi$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

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$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\psi\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,\psi = -\frac{2mr^2}{\hbar^2}(E-U)\,\psi$$

Schrodinger Equation

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$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

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$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\psi\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,\psi = -\frac{2mr^2}{\hbar^2}(E-U)\,\psi$$

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Schrodinger Equation

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Separation of variables

$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$

Schrodinger Equation

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \;[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\,(E-U)]\;\psi$$

Separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right)R\Theta\Phi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}R\Theta\Phi = \;[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\,(E-U)]\;R\Theta\Phi$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

Schrodinger Equation

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

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$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

$$R\Phi\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) + R\Theta\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\Theta\Phi\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)R\Theta\Phi$$

Schrodinger Equation

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)R\Theta\Phi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}R\Theta\Phi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]R\Theta\Phi$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

$$R\Phi\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) + R\Theta\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\Theta\Phi\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)R\Theta\Phi$$

$$\frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)$$

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-

-l (l + 1)

Angular part

Radial part

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-l (l + 1)

Angular part

Radial part

-l (l + 1)

-l (l + 1)

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-

-l (l + 1)

Angular part **-l (l + 1)**

Radial part **-l (l + 1)**

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C=-l(l+1) \text{ -} \\ \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1) \end{cases}$$

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-l (l + 1)

Angular part -l (l + 1)

Radial part -l (l + 1)

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C=-l(l+1) \\ \frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1) \end{cases}$$

$multiply\ sin^2\theta$.

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ \text{(Constant)}$$

-

-l (l + 1)

Angular part *-l (l + 1)*

Radial part *-l (l + 1)*

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C=-l(l+1)\ - \\ \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1) \end{cases}$$

$multiply\ \sin^2\theta$. $\frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-l(l+1)\sin^2\theta$

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-l (l + 1)

Angular part -l (l + 1)

Radial part -l (l + 1)

$$\begin{cases}-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U(r))=C=-l(l+1)\\ \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1)\end{cases}$$

multiply $\sin^2\theta$.
$$\frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-l(l+1)\sin^2\theta$$

$$\frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+l(l+1)\sin^2\theta=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$=m_l^2\ (\text{another constant})$$

Schrodinger Equation

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi \\ sin\theta \dfrac{\partial}{\partial \theta}\left(sin\theta \dfrac{\partial \Theta}{\partial \theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0 \\ \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial R}{\partial r}\right) + \dfrac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0 \end{cases}$$

Schrodinger Equation

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi \\ sin\theta \dfrac{\partial}{\partial \theta}\left(sin\theta \dfrac{\partial \Theta}{\partial \theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0 \\ \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial R}{\partial r}\right) + \dfrac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0 \end{cases}$$

Azimuthal Equation

Polar Equation

Radial Equation

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function

Boundary condition

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\Phi(\phi) = A\, e^{i m_l \phi}$

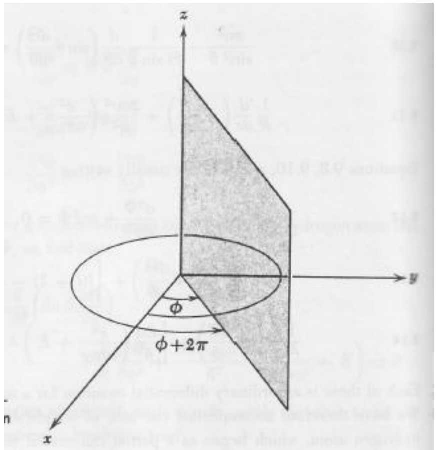
Boundary condition

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\quad \Phi(\phi) = A\, e^{i m_l \phi}$



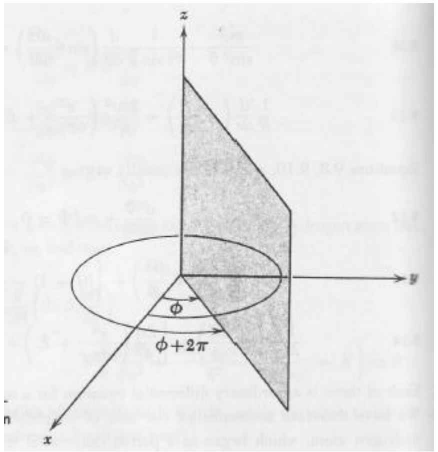
Boundary condition

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\quad \Phi(\phi) = A\, e^{i m_l \phi}$



Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

$$A\, e^{i m_l \phi} = A\, e^{i m_l(\phi + 2\pi)} = A e^{i m_l \phi} e^{i 2\pi m_l}$$

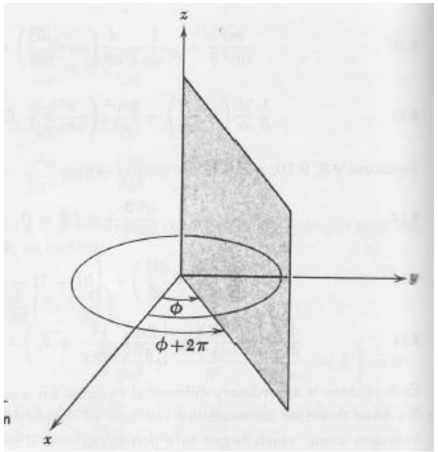
$$e^{i 2\pi m_l} = 1 = cos 2\pi m_l + \ i sin 2\pi m_l$$

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $$\Phi(\phi) = A\, e^{i m_l \phi}$$



Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

$$A\, e^{i m_l \phi} = A\, e^{i m_l (\phi + 2\pi)} = A e^{i m_l \phi} e^{i 2\pi m_l}$$

$$e^{i 2\pi m_l} = 1 = cos 2\pi m_l + \; i sin 2\pi m_l$$

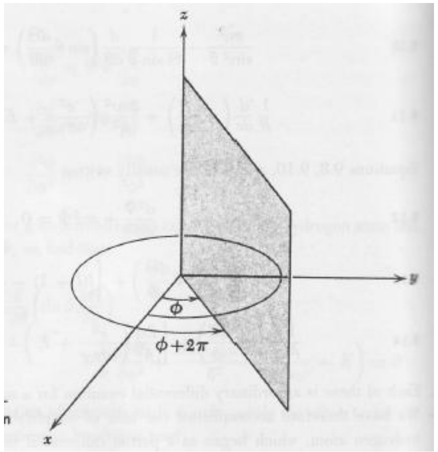
Quantization

$$m_l = 0, \pm 1, \pm 2, \pm 3, etc.$$

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\qquad \Phi(\phi) = A\, e^{i m_l \phi}$



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$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

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Quantization

$$m_l = 0, \pm 1, \pm 2, \pm 3, etc.$$

Magnetic quantum number

Polar Equation

$$sin\theta \frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0$$

Solutions: Associated Legendre Functions

Quantization

any given l, m_l *values can be* $0, \pm 1, \pm 2, \pm l$

→ **Orbital quantum number**

Radial Equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0$$

Solutions: associated Laguerre functions
Quantization:

$$E_n = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}\left(\frac{1}{n^2}\right) \text{ where } n \text{ is an integer}$$

n → **Principal quantum number**

l = 0, 1, 2, …(*n*-1)

Solutions

- Principal quantum number, $n = 1, 2, 3, \ldots$
- Orbital quantum number, $l = 0, 1, 2, \ldots (n-1)$ where l =1*(s)*, =2*(p)*, =3*(d)*, =4*(f)*, *etc.*
- Magnetic quantum number, $m_l = 0, \pm 1, \pm 2, \ldots \pm l$

$$\text{Wave function} = \psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l}$$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r,\theta,\phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$
2	1	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \cos\theta$
3	1	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4} (3\cos^2\theta - 1)$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} (3\cos^2\theta - 1)$
3	2	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2} \sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	± 2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4} \sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0^{3/2}} \frac{r^2}{a_0^2} e^{-r/3a_0} \sin^2\theta\, e^{\pm 2i\phi}$

Origin of angular momentum quantization

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi \\ sin\theta \dfrac{\partial}{\partial \theta}\left(sin\theta \dfrac{\partial \Theta}{\partial \theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0 \\ \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial R}{\partial r}\right) + \dfrac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0 \end{cases}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}\left[E - U(r) - \frac{l(l+1)\hbar^2}{2mr^2}\right]R = 0$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)R\ -l(l+1)R=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\ [E-U(r)\ -\frac{l(l+1)\hbar^2}{2mr^2}]R=0$$

$$\mathrm{E} = \text{Kinetic E (radial)} + \text{Kinetic E (orbital)} + U(r)$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,(E-U(r))R\;-l(l+1)R=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,[E-U(r)\;-\frac{l(l+1)\hbar^2}{2mr^2}]R=0$$

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$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,[Kineric\ E\ (radial)+\ Kinetic\ E\ (orbital)\;-\frac{l(l+1)\hbar^2}{2mr^2}]R=0$$

$$Kinetic\ E\ (orbital)=\frac{l(l+1)\hbar^2}{2mr^2}$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)R-l(l+1)R=0$$

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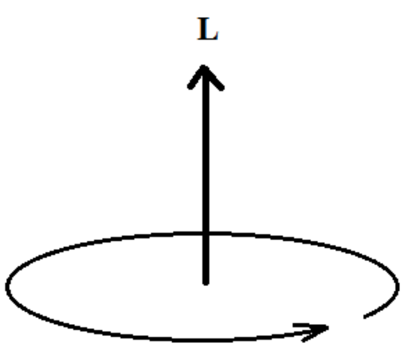
$$\text{E} = \text{Kinetic E (radial)} + \text{Kinetic E (orbital)} + U(r)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left[Kineric\ E\ (radial) + Kinetic\ E\ (orbital) - \frac{l(l+1)\hbar^2}{2mr^2}\right]R=0$$

$$Kinetic\ E\ (orbital) = \frac{l(l+1)\hbar^2}{2mr^2}$$



$$\frac{1}{2}m\,v_{orbital}^2 = \frac{L^2}{2mr^2}$$

$$L = mv_{orbital}r \rightarrow v_{orbital} = \frac{L}{mr}$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)R-l(l+1)R=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left[E-U(r)-\frac{l(l+1)\hbar^2}{2mr^2}\right]R=0$$

$$\text{E} = \text{Kinetic E (radial)} + \text{Kinetic E (orbital)} + U(r)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left[Kineric\ E\ (radial)+\ Kinetic\ E\ (orbital)-\frac{l(l+1)\hbar^2}{2mr^2}\right]R=0$$

$$Kinetic\ E\ (orbital)=\frac{l(l+1)\hbar^2}{2mr^2}$$

$$\frac{1}{2}m\,v_{orbital}^2=\frac{L^2}{2mr^2}$$

$$L=mv_{orbital}r \rightarrow v_{orbital}=\frac{L}{mr}$$

$$\frac{L^2}{2mr^2}=\frac{l(l+1)\hbar^2}{2mr^2}$$

$$L^2=\ l(l+1)\hbar^2$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,(E-U(r))R\;-l(l+1)R=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,[E-U(r)\;-\frac{l(l+1)\hbar^2}{2mr^2}]R=0$$

$$\text{E} = \text{Kinetic E (radial)} + \text{Kinetic E (orbital)} + U(r)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,[Kineric\ E\ (radial)+\ Kinetic\ E\ (orbital)\;-\frac{l(l+1)\hbar^2}{2mr^2}]R=0$$

$$Kinetic\ E\ (orbital)=\frac{l(l+1)\hbar^2}{2mr^2}$$

$$\frac{1}{2}m\,v_{orbital}^2=\frac{L^2}{2mr^2}$$

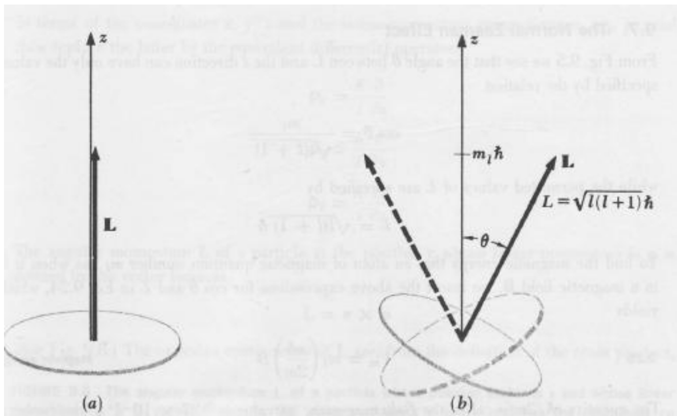
$$L=mv_{orbital}r \rightarrow v_{orbital}=\frac{L}{mr}$$

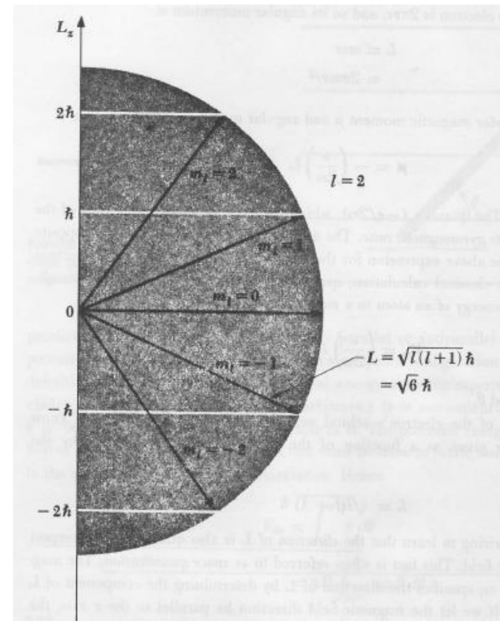
$$\frac{L^2}{2mr^2}=\frac{l(l+1)\hbar^2}{2mr^2}$$

$$L^2=\;l(l+1)\hbar^2$$

Angular momentum is quantized

Angular Momentum (L)

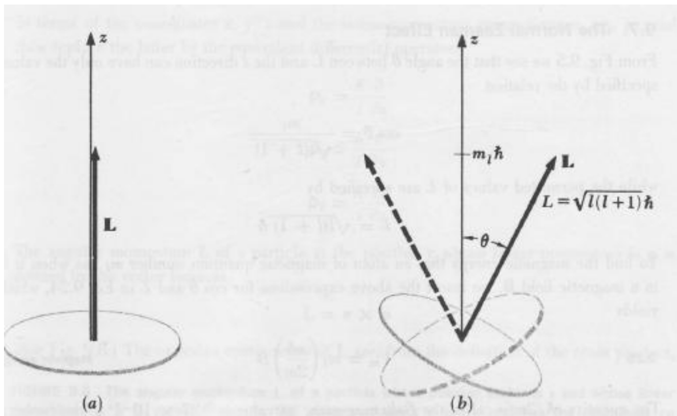


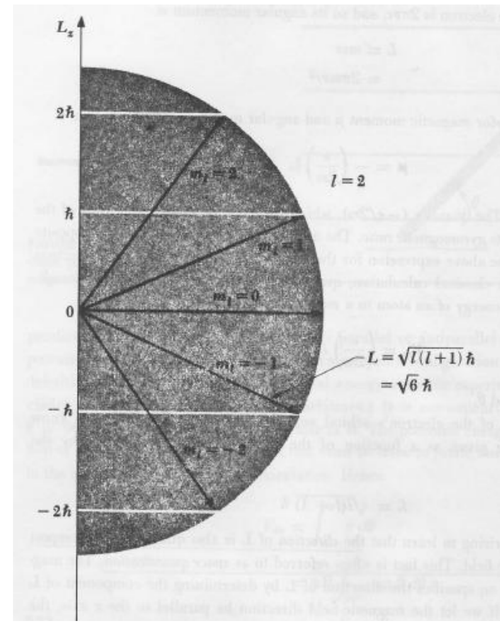


Amount =

Direction =

Angular Momentum (L)

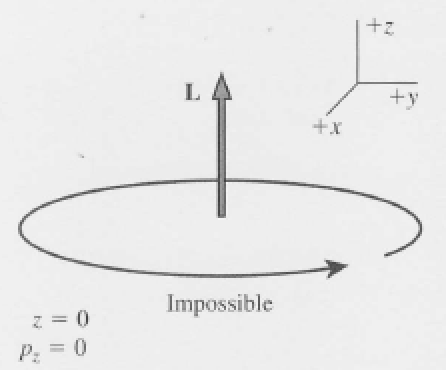




Amount = $L = \sqrt{l(l+1)}\hbar$

Direction = the m_l *value* determines the direction of L

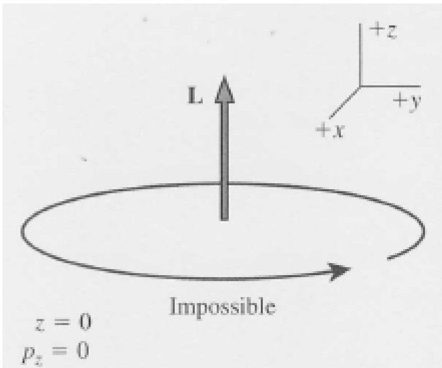
Lz and Uncertainty principle

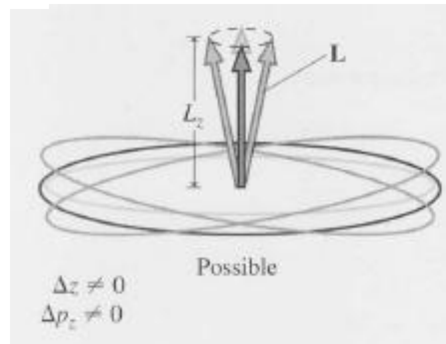


$$\Delta z \Delta p_z > \frac{\hbar}{2}$$

Lz and Uncertainty principle

$$\Delta z \Delta p_z > \frac{\hbar}{2}$$





Schrodinger Equation: Hydrogen Atom

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x}) \qquad U(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + U(\vec{x})\right]\psi(\vec{x}) = E\,\psi(\vec{x})$$

Hamiltonian (H) = T (kinetic) + U (Potential)

Kinetic energy w.r.t. r + Kinetic energy w.r.t. rotation

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(E - U(r)\right) = C = -l(l+1) \\ \frac{1}{\Theta}\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\csc^2\theta\frac{\partial^2\Phi}{\partial\phi^2} = C = -l(l+1) \end{cases}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = -m_l^2\Phi$$

Schrodinger Equation: Hydrogen Atom

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x}) \qquad U(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

$$[\frac{-\hbar^2}{2m}\nabla^2 + U(\vec{x})]\,\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\mathbf{H}\psi_{n,l,m_l} = E_n\psi_{n,l,m_l}$$

Hamiltonian (H) = T (kinetic) + U (Potential)

Kinetic energy w.r.t. r + Kinetic energy w.r.t. rotation

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(E - U(r)\right) = C = -l(l+1) \\ \frac{1}{\Theta}\csc\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2} = C = -l(l+1) \end{cases}$$

$$\mathbf{L}^2\psi_{n,l,m_l} = l(l+1)\hbar^2\psi_{n,l,m_l}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = -m_l^2\Phi$$

$$\mathbf{L_z}\psi_{n,l,m_l} = m_l\hbar\psi_{n,l,m_l}$$

Three quantum numbers

- Principal quantum number: *n*

$$E_n = -\frac{me^4}{32\pi^2 \varepsilon_0^2 \hbar^2}\left(\frac{1}{n^2}\right) = -\left(\frac{e^2}{8\pi\varepsilon_0}\right)\left(\frac{me^2}{4\pi\varepsilon_0 \hbar^2}\right)\left(\frac{1}{n^2}\right) =$$
$$-\left(\frac{e^2}{8\pi\varepsilon_0 a_0}\right)\left(\frac{1}{n^2}\right) = -13.6\ eV\left(\frac{1}{n^2}\right)$$

$$L^2 = l(l+1)\hbar^2$$
$$L_z = m_l \hbar$$

- Orbital quantum number: $l = 0, 1, 2, \ldots (n\text{-}1)$
- Magnetic quantum number: $m_l = 0, \pm 1, \pm 2, \ldots \pm l$

Degeneracies

$$\psi(r,\theta,\phi) = \psi_{n,l,m_l} = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l} = R_{n,l}\mathbf{Y}_l^{m_l}$$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

n	l	m_l	E_n(eV)	$\lvert L\rvert$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s
2	0	0	-3.40	0	0	ψ_{200}	$R_{20}\mathbf{Y}_0^0$	4 (=2^2)	2s
	1	-1		$\sqrt{2}\hbar$	$-\hbar$	ψ_{21-1}	$R_{21}\mathbf{Y}_1^{-1}$		2p
		0			0	ψ_{210}	$R_{21}\mathbf{Y}_1^0$		
		1			$+\hbar$	ψ_{211}	$R_{21}\mathbf{Y}_1^{+1}$		

n	l	m_l	E_n(eV)	$\|L\|$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s
2	0	0	-3.40	0	0	ψ_{200}	$R_{20}\mathbf{Y}_0^0$	4 (=2^2)	2s
	1	-1		$\sqrt{2}\hbar$	$-\hbar$	ψ_{21-1}	$R_{21}\mathbf{Y}_1^{-1}$		2p
		0			0	ψ_{210}	$R_{21}\mathbf{Y}_1^0$		
		1			$+\hbar$	ψ_{211}	$R_{21}\mathbf{Y}_1^{+1}$		
3	0	0	-1.51	0	0	ψ_{300}	$R_{30}\mathbf{Y}_0^0$	9 (=3^2)	3s
	1	-1		$\sqrt{2}\hbar$	$-\hbar$	ψ_{31-1}	$R_{31}\mathbf{Y}_1^{-1}$		3p
		0			0	ψ_{310}	$R_{31}\mathbf{Y}_1^0$		
		1			$+\hbar$	ψ_{311}	$R_{31}\mathbf{Y}_1^1$		
	2	-2		$\sqrt{6}\hbar$	$-2\hbar$	ψ_{32-2}	$R_{32}\mathbf{Y}_2^{-2}$		3d
		--1			$-\hbar$	ψ_{32-1}	$R_{32}\mathbf{Y}_2^{-1}$		
		0			0	ψ_{320}	$R_{32}\mathbf{Y}_2^0$		
		1			$+\hbar$	ψ_{321}	$R_{32}\mathbf{Y}_2^1$		
		2			$+2\hbar$	ψ_{322}	$R_{32}\mathbf{Y}_2^2$		

Symbolic designation of atomic states

	s *l* = 0	*p* *l* =1	*d* *l* =2	*f* *l* =3	*g* *l* = 4	*h* *l* =5
n = 1	*1s*					
n = 2	*2s*	*2p*				
n = 3	*3s*	*3p*	*3d*			
n = 4	*4s*	*4p*	*4d*	*4f*		
n = 5	*5s*	*5p*	*5d*	*5f*	*5g*	
n = 6	*6s*	*6p*	*6d*	*6f*	*6g*	*6h*

Normalization

$$|\psi_{n,l,m_l}|^2 = R(r)^2\Theta(\theta)^2\Phi(\phi)^2$$

$$\int |\psi_{n,l,m_l}|^2\, dV = \int |\psi_{n,l,m_l}|^2\, r^2 sin\theta\; dr d\theta d\phi$$

$$= \int_0^\infty R(r)^2 r^2\, dr \int_0^\pi \Theta(\theta)^2 sin\theta\; d\theta \int_0^{2\pi} \Phi(\phi)^2 d\phi$$

$$\begin{cases} \int_0^\infty R(r)^2 r^2\, dr = 1 \\ \int_0^\pi \Theta(\theta)^2 sin\theta\; d\theta \int_0^{2\pi} \Phi(\phi)^2 d\phi = 2\pi \int_0^\pi \Theta(\theta)^2 sin\theta\; d\theta = 1 \end{cases}$$

Angular Probability Density

$$\Theta(\theta)^2\Phi(\phi)^2 \equiv \mathbf{Y}_l^{m_l *}\mathbf{Y}_l^{m_l}$$

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r,\theta,\phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0^{3/2}}e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\,a_0^{3/2}}e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\,a_0^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\,a_0^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{1}{2\sqrt{6}\,a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\,a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}\cos\theta$
2	1	±1	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{2\sqrt{6}\,a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\,a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}\sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\,a_0^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0^2}\right)e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\,a_0^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0^2}\right)e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{4}{81\sqrt{6}\,a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\,a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}e^{-r/3a_0}\cos\theta$
3	1	±1	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{4}{81\sqrt{6}\,a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\,a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}e^{-r/3a_0}\sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4}(3\cos^2\theta-1)$	$\frac{4}{81\sqrt{30}\,a_0^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\,a_0^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}(3\cos^2\theta-1)$
3	2	±1	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$	$\frac{\sqrt{15}}{2}\sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\,a_0^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\,a_0^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}\sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	±2	$\frac{1}{\sqrt{2\pi}}e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\frac{4}{81\sqrt{30}\,a_0^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\,a_0^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}\sin^2\theta\, e^{\pm 2i\phi}$

Angular Probability Density

$$\Theta(\theta)^2\Phi(\phi)^2 \equiv \mathbf{Y}_l^{m_l *}\mathbf{Y}_l^{m_l}$$

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r,\theta,\phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0^{3/2}}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0^{3/2}}\frac{r}{a_0} e^{-r/2a_0}\cos\theta$
2	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{2\sqrt{6}\, a_0^{3/2}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0^{3/2}}\frac{r}{a_0} e^{-r/2a_0}\sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\cos\theta$
3	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{4}{81\sqrt{6}\, a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4}(3\cos^2\theta-1)$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}(3\cos^2\theta-1)$
3	2	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2}\sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}\sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	±2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0^{3/2}}\frac{r^2}{a_0^2} e^{-r/3a_0}\sin^2\theta\, e^{\pm 2i\phi}$

Unsöld’s theorem

$$\sum_{m_l=-l}^{l} |\Theta_{l,m_l}|^2 |\Phi_{m_l}|^2 = Constant$$

Angular Probability Density

$l = 0,\ m_l = 0, \quad \Theta_{00} = \frac{1}{\sqrt{2}}, \Phi_0 = \frac{1}{\sqrt{2\pi}}$

$l = 1, m_l = 0, \quad \Theta_{10} = \frac{\sqrt{6}}{2}\cos\theta, \Phi_0 = \frac{1}{\sqrt{2\pi}}$

$l = 1, m_l = \pm 1, \quad \Theta_{1\pm1} = \frac{\sqrt{3}}{2}\sin\theta, \Phi_{\pm1} = \frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$

$l = 2, m_l = 0, \quad \Theta_{20} = \frac{\sqrt{10}}{4}(3\cos^2\theta - 1), \Phi_0 = \frac{1}{\sqrt{2\pi}}$

$l = 2, m_l = \pm 1, \quad \Theta_{2\pm1} = \frac{\sqrt{15}}{2} sin\theta cos\theta, \Phi_{\pm1} = \frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$

$l = 2, m_l = \pm 2, \quad \Theta_{2\pm2} = \frac{\sqrt{15}}{4} sin^2\theta, \quad \Phi_{\pm2} = \frac{1}{\sqrt{2\pi}}e^{\pm 2i\phi}$

Unsöld's theorem

$$\sum_{m_l=-l}^{l} |\Theta_{l,m_l}|^2 |\Phi_{m_l}|^2 = Constant$$

Angular Probability Density

$l = 0,\ m_l = 0, \quad \Theta_{00} = \frac{1}{\sqrt{2}}, \Phi_0 = \frac{1}{\sqrt{2\pi}}$

$l = 1, m_l = 0, \quad \Theta_{10} = \frac{\sqrt{6}}{2}\cos\theta, \Phi_0 = \frac{1}{\sqrt{2\pi}}$

$l = 1, m_l = \pm 1, \quad \Theta_{1\pm1} = \frac{\sqrt{3}}{2}\sin\theta, \Phi_{\pm1} = \frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$

$l = 2, m_l = 0, \quad \Theta_{20} = \frac{\sqrt{10}}{4}(3\cos^2\theta - 1), \Phi_0 = \frac{1}{\sqrt{2\pi}}$

$l = 2, m_l = \pm 1, \quad \Theta_{2\pm1} = \frac{\sqrt{15}}{2} sin\theta cos\theta, \Phi_{\pm1} = \frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$

$l = 2, m_l = \pm 2, \quad \Theta_{2\pm2} = \frac{\sqrt{15}}{4} sin^2\theta, \quad \Phi_{\pm2} = \frac{1}{\sqrt{2\pi}}e^{\pm 2i\phi}$

Unsöld's theorem

$$\sum_{m_l = -l}^{l} |\Theta_{l,m_l}|^2 |\Phi_{m_l}|^2 = Constant$$

For $l = 0$,

$$\sum_{m_l = 0}^{0} |\Theta_{0,m_l}|^2 |\Phi_{m_l}|^2 = {\Theta_{00}}^2 {\Phi_0}^2 = \frac{1}{4\pi} = constant$$

Angular Probability Density

$$\Theta(\theta)^2\Phi(\phi)^2 \equiv \mathbf{Y}_l^{m_l *}\mathbf{Y}_l^{m_l}$$

Figure 7.13 Angular probability densities for a central force.

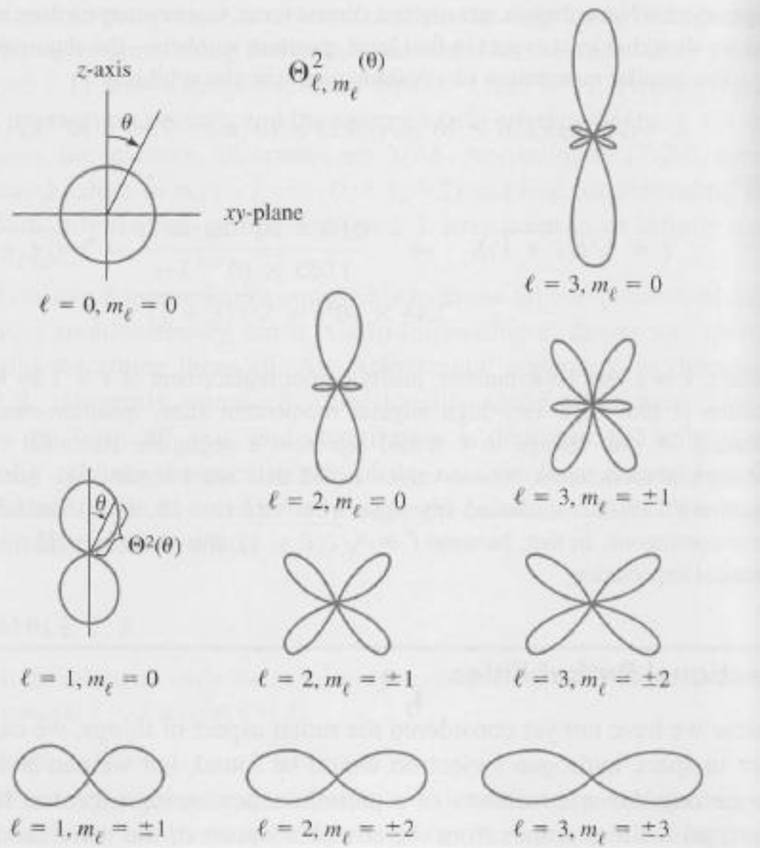


Figure 7.14 A crude correspondence to orbital motion.