

PH102: Modern Physics Homework 2 (Due: 1/18/2013)

1. (10 points) A particle of mass m in a 2 dimensional infinite well L long and W wide.

- (a) Write the time-independent Schrodinger Equation for a particle in the two dimensional infinite well.
- (b) Solve the Schrodinger Equation and obtain allowed energy values and associated wave functions.
- (c) Obtain the ground state energy level and the associated wave function. Obtain the probability density related to the ground state. Sketch the probability density.
- (d) Consider the special case $L = W$. Find the five lowest energy levels of a particle in this square well. Which of these levels is degenerate?

2. (10 points) Angular momentum

The angular momentum \mathbf{L} of a particle at the position \mathbf{r} whose linear momentum is \mathbf{p} is defined by the vector formula

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

- (a) From the above expression, explain how three Cartesian components of \mathbf{L} can be written as:

$$\begin{cases} L_x = yp_z - zp_y \\ L_y = zp_x - xp_z \\ L_z = xp_y - yp_x \end{cases}$$

- (b) Using the momentum operators such as $p_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$, etc. express three Cartesian angular momentum operators L_x, L_y, L_z .

- (c) Prove that the three Cartesian angular momentum operators in spherical polar coordinates can be written as:

$$\begin{aligned} L_x &= \frac{\hbar}{i} \left(-\sin\phi \frac{\partial}{\partial \theta} - \cot\theta \cos\phi \frac{\partial}{\partial \phi} \right) \\ L_y &= \frac{\hbar}{i} \left(\cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi} \right) \\ L_z &= \frac{\hbar}{i} \frac{\partial}{\partial \phi} \end{aligned}$$

- (d) L^2 operator is defined by:

$$L^2 = L_x L_x + L_y L_y + L_z L_z$$

Prove that

$$\mathbf{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

(e) (Optional) Using $L_z = \frac{\hbar}{i} \frac{\partial}{\partial\phi}$ and $\Phi(\phi) = A e^{im_l\phi}$ prove that

$$L_z\psi = m_l\hbar\psi$$

(f) (Optional) Using $\mathbf{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$

$$\Phi(\phi) = A e^{im_l\phi}$$

$$\sin\theta \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\theta}{\partial\theta} \right) + [l(l+1)\sin^2\theta - m_l^2]\theta = 0$$

Prove that

$$\mathbf{L}^2 \psi = l(l+1)\hbar^2\psi$$

3. (10 points) Textbook: Harris, Chapter 7 Conceptual Question #8
4. (10 points) Textbook: Harris, Chapter 7 Conceptual Question #12
5. (10 points) Textbook: Harris, Chapter 7 Exercises Question #39
6. (10 points) Textbook: Harris, Chapter 7 Exercises Question #70

[EXTRA STUDY]

Below is a set of questions used in the last year's homework.

They are NOT your homework. However, study these problems to prepare for exams as they are important and interesting. The solutions are available in my teaching website.

<https://griffin.ucsc.edu/ph102-12/Homework%2B>

1. Textbook: Harris, Chapter 7 Conceptual Question #7
2. Textbook: Harris, Chapter 7 Conceptual Question #9
3. Textbook: Harris, Chapter 7 Conceptual Question #14
4. Unsöld's theorem states that, for any value of the orbital quantum number l , the probability densities summed over all possible states from $m_l = -l$ to $m_l = +l$ yield a constant independent of angles θ or ϕ , that is

$$\sum_{m_l=-l}^l |\Theta_{l,m_l}|^2 |\Phi_{m_l}|^2 = \text{Constant}$$

- (a) Verify Unsöld's theorem for $l = 0$, $l = 1$, and $l = 2$.
 - (b) Explain what this theorem tells about the charge distribution of the closed shell atoms such as Helium and Argon.
5. Let us consider the angular wave functions for $l = 1$. We leave n as unspecified and general: it is a fixed, but arbitrary, positive integer. Thus, this problem applies to $2p$ orbitals, $3p$ orbitals, or $4p$ orbitals, etc., regardless of the fixed value of n .

(Note: $\Theta_{l,m_l} \Phi_{m_l} = Y_l^{m_l}$, spherical harmonics).

- (a) Show that the angular wave function Y_1^0 has the same angular dependence (θ and ϕ) as the coordinate z expressed in terms of the spherical coordinates (r, θ, ϕ) .
- (b) Show that by proper linear combinations of Y_1^1 and Y_1^{-1} , it is possible to construct two wave functions that have the same angular dependence as the coordinates x and y respectively. Let us call these two new wave functions $Y_x(\theta, \phi)$ and $Y_y(\theta, \phi)$.
- (c) Show that Y_x , Y_y , and Y_z (where $Y_z = Y_1^0$) are mutually orthogonal to each other.

Note that this new set of vectors, Y_x , Y_y , and Y_z , also provide a complete description of p orbitals, just as the original $Y_l^{m_l}$'s. This new set may be familiar to you from chemistry textbooks: they are the so-called p_x , p_y , and p_z orbitals. While the two different sets for p orbitals, $\{p_x, p_y, p_z\}$ and $\{Y_1^1, Y_1^{-1}, Y_1^0\}$, are completely equivalent for describing atomic orbitals, the $\{p_x, p_y, p_z\}$ set provides a definitely more natural way to describe p orbitals in chemical bonds.

To test the orthogonality among Y_x , Y_y , and Y_z , use the following:

- Orthogonality between Y_x , Y_y , check whether $\int_0^\pi \int_0^{2\pi} Y_x^* Y_y \sin\theta d\theta d\phi = 0$.
- Orthogonality between Y_y , Y_z , check whether $\int_0^\pi \int_0^{2\pi} Y_y^* Y_z \sin\theta d\theta d\phi = 0$.
- Orthogonality between Y_x , Y_z , check whether $\int_0^\pi \int_0^{2\pi} Y_x^* Y_z \sin\theta d\theta d\phi = 0$.