

PH102, 2012W, Lecture Notes: January 29, Mon, Class 9

Angular Momentum in External Magnetic Fields

An electron revolving around a nucleus in an atom has two types of angular momentum:

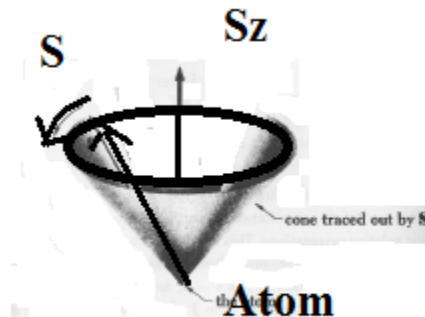
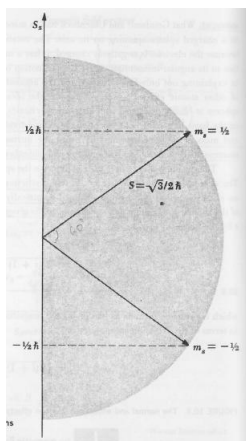
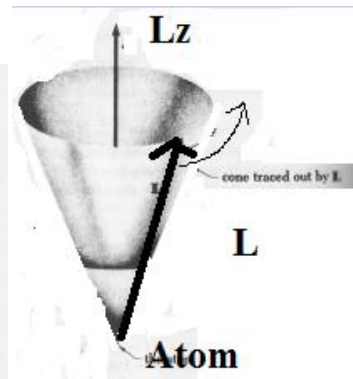
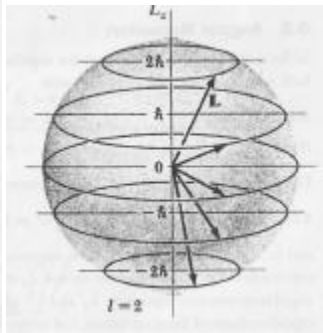
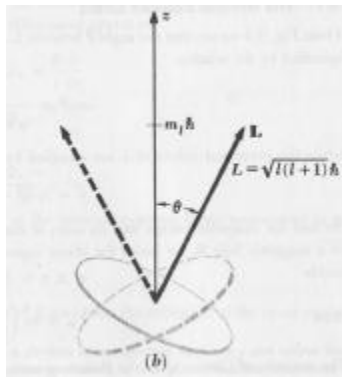
- Orbital angular momentum:  $\vec{L}$
- Spin angular momentum:  $\vec{S}$

$$|\vec{L}| = \sqrt{l(l+1)}\hbar \quad \text{where } l = 0, 1, 2, \dots, n-1$$

$$L_z = m_l \hbar \quad \text{where } m_l = -l, -l+1, \dots, l-1, l$$

$$|\vec{S}| = \sqrt{s(s+1)}\hbar \quad \text{where } s \text{ is a number intrinsic to a given particle}$$

$$S_z = m_s \hbar \quad \text{where } m_s = -s, -s+1, \dots, s-1, s$$

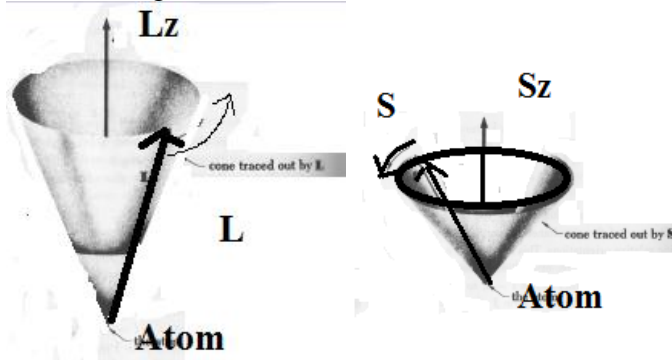


In the presence of an external magnetic field,  $\vec{B}$ , we need to consider

- Magnetic dipole moment related to orbital angular momentum of an electron,  $\vec{\mu}_L = -\left(\frac{e}{2m}\right)\vec{L}$
- Magnetic dipole moment related to spin angular momentum of an electron,  $\vec{\mu}_S = -\left(\frac{e}{m}\right)\vec{S}$

**The Paschen-Back Effect:**

When the external magnetic field  $\vec{B}$  is strong, then the  $\vec{L}$  and  $\vec{S}$  are separately aligned with the external magnetic field.



Energy related to the magnetic moment due to  $\vec{L}$

$$U_L = -\vec{\mu}_L \cdot \vec{B} = -\mu_{L_z} B_z = -\left(-\frac{e}{2m} L_z\right) B_z = \frac{e}{2m} m_l \hbar B_z$$

Energy related to the magnetic moment due to  $\vec{S}$

$$U_S = -\vec{\mu}_S \cdot \vec{B} = -\mu_{S_z} B_z = -\left(-\frac{e}{m} S_z\right) B_z = \frac{e}{m} m_s \hbar B_z$$

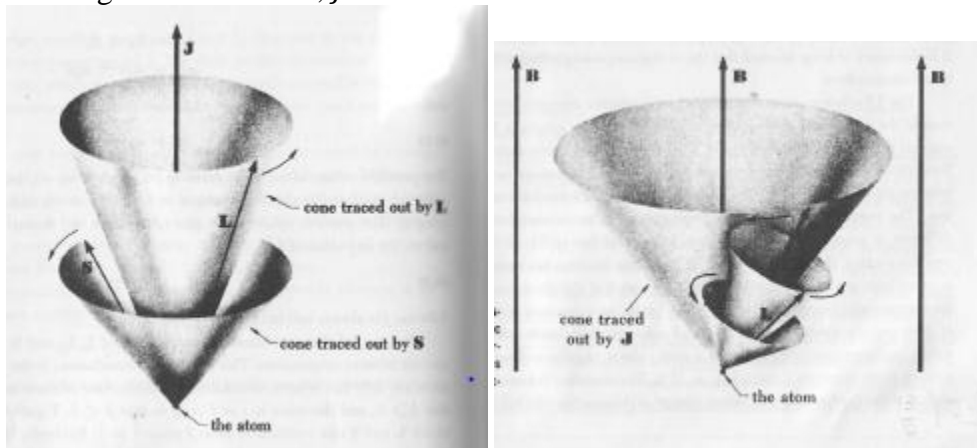
Therefore  $E_n$  will split into:

$$E_{strong\ magnetic\ field} = E_n + \frac{e}{2m} (m_l + 2m_s) \hbar B_z$$

In this case,  $n, l, m_l, m_s$  are good quantum numbers.

**The Zeeman Effect:**

When the external magnetic field  $\vec{B}$  is weak, then the  $\vec{L}$  and  $\vec{S}$  of an electron are coupled into a total angular momentum,  $\vec{J} = \vec{L} + \vec{S}$



Therefore, the total magnetic moment can be written as

$$\vec{\mu}_J = \vec{\mu}_L + \vec{\mu}_S = -\left(\frac{e}{2m}\right)(\vec{L} + 2\vec{S})$$

Note that  $\vec{J} = \vec{L} + \vec{S}$  and  $\vec{\mu}_J = -\left(\frac{e}{2m}\right)(\vec{L} + 2\vec{S})$  are not aligned. We need to project the  $\vec{\mu}_J$  onto the  $\vec{J}$ . Consider the angle between  $\vec{\mu}_J$  and  $\vec{J}$  as  $\pi + \delta$ .

Then,  $|\vec{\mu}_J||\vec{J}|\cos(\pi + \delta) = \vec{\mu}_J \cdot \vec{J}$  and  $\cos \delta = -\frac{\vec{\mu}_J \cdot \vec{J}}{|\vec{\mu}_J||\vec{J}|}$

$$\begin{aligned} |\vec{\mu}_J| &= |\vec{\mu}_J| \cos \delta = |\vec{\mu}_J| \left( -\frac{\vec{\mu}_J \cdot \vec{J}}{|\vec{\mu}_J||\vec{J}|} \right) = -\frac{\vec{\mu}_J \cdot \vec{J}}{|\vec{J}|} = \frac{e}{2m} \frac{(\vec{L} + 2\vec{S}) \cdot (\vec{L} + \vec{S})}{|\vec{J}|} \\ &= \frac{e}{2m} \frac{|\vec{L}|^2 + 2|\vec{S}|^2 + 3\vec{L} \cdot \vec{S}}{|\vec{J}|} = \end{aligned}$$

$$\text{since } \vec{L} \cdot \vec{S} = \frac{1}{2}(|\vec{J}|^2 - |\vec{L}|^2 - |\vec{S}|^2)$$

$$|\vec{\mu}_J| = \frac{e}{2m} \frac{3|\vec{J}|^2 - |\vec{L}|^2 + |\vec{S}|^2}{2|\vec{J}|} = \frac{e}{2m} \frac{3j(j+1) - l(l+1) + s(s+1)}{2\sqrt{j(j+1)}} \hbar$$

Consider  $\vec{B} = B_z \hat{z}$ , then potential energy and since  $\cos \theta = \frac{J_z}{|\vec{J}|} = \frac{J_z}{\sqrt{j(j+1)}\hbar}$

$$\begin{aligned} U = -\vec{\mu}_J \cdot \vec{B} &= |\vec{\mu}_J| B_z \cos \theta = |\vec{\mu}_J| B_z \frac{J_z}{\sqrt{j(j+1)}\hbar} = \frac{e J_z B_z}{2m} \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} = \\ &= g_{Lande} \frac{e J_z B_z}{2m} = g_{Lande} \frac{e B_z}{2m} (m_j \hbar) = \end{aligned}$$

$$\text{Where } g_{Lande} = \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$$

Note that

- if  $s=0$  and  $j=l$ ,  $g_{Lande} = 1$
- If  $l=0$  and  $j=s$ ,  $g_{Lande} = 2$
- Therefore, the degenerate  $E_n$  of the hydrogen atom is modified under a weak magnetic field as
- $E_{\text{weak external magnetic field}} = E_n + g_{Lande} \frac{e B_z}{2m} (m_j \hbar)$
- In this case,  $n, l, j, m_j$  are good quantum numbers

Then, what counts as a strong and weak external magnetic field? We use the interaction (L and S) between the magnetic field generated by the orbiting electron and the electron's spin.

$$U = -\vec{\mu}_S \cdot \vec{B}_{\text{due to } \vec{L}}$$

$$\vec{B} = \frac{\mu_0 e}{4\pi m r^3} \vec{L}$$

For  $2p$  state,  $l = 1, n = 2$ , and  $r = n^2 a_0$

$$B = \frac{\mu_0 e}{4\pi m (4a_0)^3} \sqrt{1} \cdot 2\hbar = 0.28 \text{ T}$$

Associated energy level  $U = \mu_S B = 2 \times 10^{-5} \text{ eV}$

**Therefore, the energy associated with LS coupling is very small.**

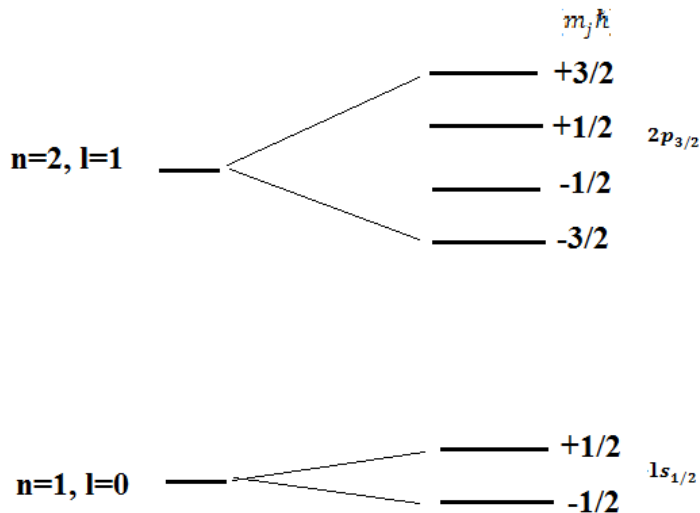
**A magnetic field that can interrupt the LS coupling, therefore, should be much larger than .28T.**

**Example :  $2p_{3/2} \rightarrow 1s_{1/2}$  in a weak external magnetic field of .05T**

For  $1s_{1/2}, l = 0, s = \frac{1}{2}, j = \frac{1}{2}; m_j = -\frac{1}{2}, \frac{1}{2}$

For  $2p_{3/2}, l = 1, s = \frac{1}{2}, j = \frac{3}{2}; m_j = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$

The split due to LS coupling is shown in the diagram:



We can calculate the amount of deviation occurs from the  $E_n$  value in each orbital by first calculating  $g_{Lande}$

$$g_{Lande} = \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$$

For  $1s_{1/2}, l = 0, s = \frac{1}{2}, j = \frac{1}{2}; g_{Lande} = 2;$

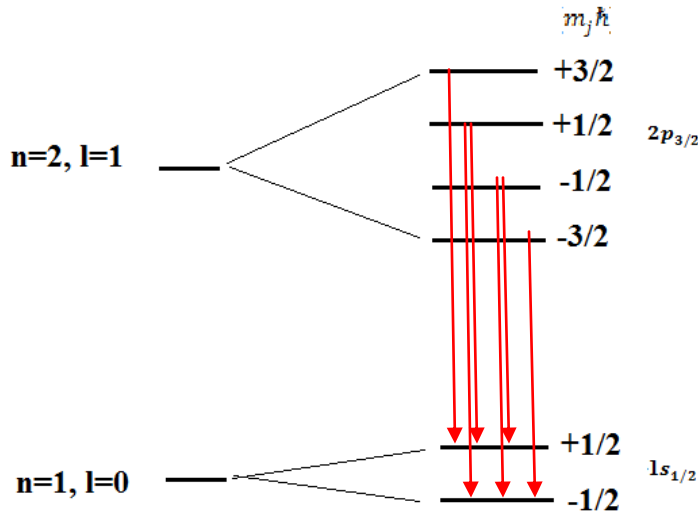
$$\Delta E = g_{Lande} \frac{eB_z \hbar}{2m} (m_j \hbar) = \frac{2eB_z \hbar}{2m} (m_j) = \frac{2eB_z \hbar}{2m} \left( \pm \frac{1}{2} \right)$$

For  $2p_{3/2}, l = 1, s = \frac{1}{2}, j = \frac{3}{2}; g_{Lande} = \frac{4}{3}$

$$\Delta E = g_{Lande} \frac{eB_z}{2m} (m_j \hbar) = \left(\frac{4}{3}\right) \frac{eB_z \hbar}{2m} (m_j) = \frac{2eB_z \hbar}{3m} \left(\pm \frac{3}{2}, \pm \frac{1}{2}\right)$$

Transition between the two states can occur when  $\Delta m_j = 0, \pm 1$  (because photon's spin is 1 and the total angular momentum should be conserved before and after the transition).

This means that only the following transitions are allowed:



Therefore, a photon can have the energy difference:

$\Delta E = \Delta E_{without\ magnetic\ field} + \frac{2eB_z \hbar}{2m} \left(\pm \frac{5}{3}, \pm \frac{3}{3}, \pm \frac{1}{3}\right) \rightarrow$  therefore 1 spectral line splits into 6 spectral lines in the presence of a weak magnetic field. The energy order of the split is  $1.9 \times 10^{-6} eV$ . Compare this with the difference between  $E_2 - E_1 = -13.6 \left(\frac{1}{4} - 1\right) eV = 10.2 eV$ .