PH102, 2012W, Lecture Notes: February 6, Mon, Class 11

Probability Distributions: Maxwell-Boltzman, Fermi-Dirac, Bose-Einstein

Three different types of distributions

Consider that *k* energy states in an increasing order are available for the particles to occupy and $\mathcal{N}(E_i)$ represents the number of particles occupying the *i*th energy state. All systems should satisfy two conditions:

- Conservation of particles: $\sum \mathcal{N}(E_i) = \mathcal{N}(E_1) + \mathcal{N}(E_2) + \cdots + \mathcal{N}(E_k) = N$
- Conservation of energy: $\sum \mathcal{N}(E_i)E_i = \mathcal{N}(E_1)E_1 + \mathcal{N}(E_2)E_2 + \cdots + \mathcal{N}(E_k)E_k = E$

In order to compare differences among three types of probability distributions, consider systems of four harmonic oscillators where $E_i = n_i\hbar\omega_0$ and n_i represents a quantum number associated with the energy state of the *i*th particle, which starts from 0, 1,…).

Consider the total energy of the system is $2\hbar\omega_0$.

First, when the particles (so we label particles a, b, c, d) are classical, which means they are distinguishable.

- Conservation of particles: the total number of particles is 4.
  Conservation of energy can be rewritten in terms of quantum numbers associated with the four particles: $n_a + n_b + n_c + n_d = 2$

  Ways of distributing $2\hbar\omega_0$over four distinguishable particles:

n	10 possible ways are possible										No. of possibilities where a particle can have n quantum number (#)	Probability of a particle having the n quantum number (P=#/40)	Probable number of particles to have the n quantum number P x 4
2	*a*	*b*	*c*	*d*							4	0.1	0.4
1					*ab*	*ac*	*ad*	*bc*	*bd*	*cd*	12	0.3	1.2
0	*bcd*	*acd*	*abd*	*abc*	*cd*	*bd*	*bc*	*ad*	*ac*	*ab*	24	0.6	2.4
										Total	40	1.0	4.0

If the four particles are BOSONs that can occupy the same quantum energy state and are indistinguishable from each other, then there are only two possible ways since ways 1 to 4 are the same and ways 5 to 10 are the same:

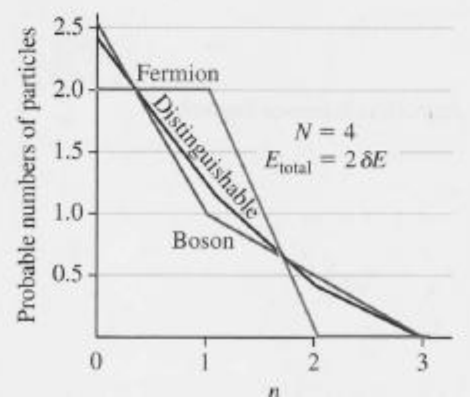
n	Two possible ways		(#)	(P=#/8)	P x 4
2	X		1	0.125	0.50
1		XX	2	0.250	1.00
0	XXX	XX	5	0.625	2.50
		Total	8	1.000	4.00

If the four particles are FERMIONs, then more than two particles cannot occupy the same quantum energy state. There is only one possible way:

n	1 possible way	(#)	(P=#/8)	P x 4
2		0	0.0	0.0
1	XX	2	0.5	2.0
0	XX	2	0.5	2.0
	Total	4	1.0	4.0

- In all three types of distributions, the probable number of particles in a given state decreases with increasing energy.
- Bosons like to be in the ground state more than the classical particles and Fermions less like to be in the ground state. Compare probable number for the ground state (Fermion,2.0) < (Classical, 2.4) < (Boson, 2.5)
- If we draw a graph for the probable number over *n*:

Figure 9.9 The probable number of particles at the allowed energies depends on whether the particles are bosons, fermions, or distinguishable.

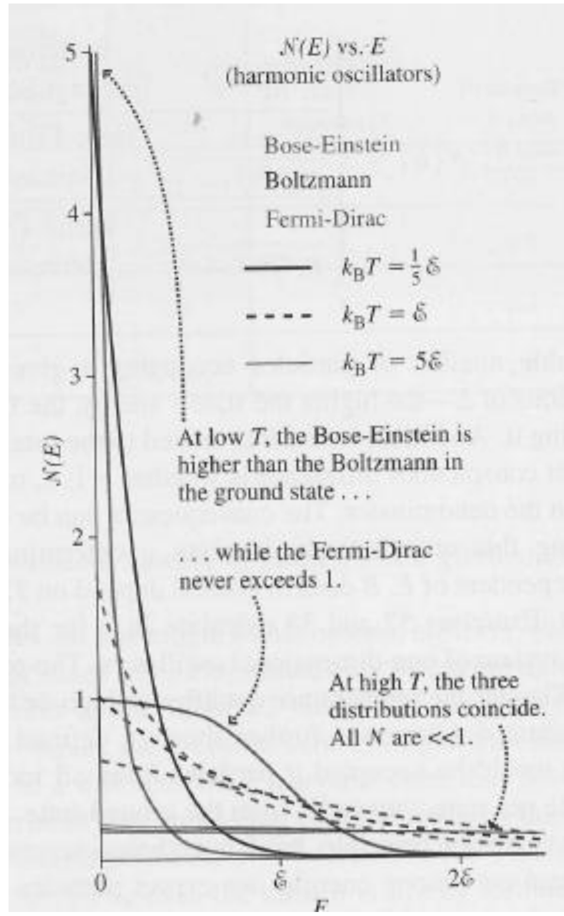


How do Boltzman, Bose-Einstein, and Fermi-Dirac distributions compare?

When N is large, the exact statistical probability calculations can be replaced with the three types of probability distributions depending upon the type of particles in a system.

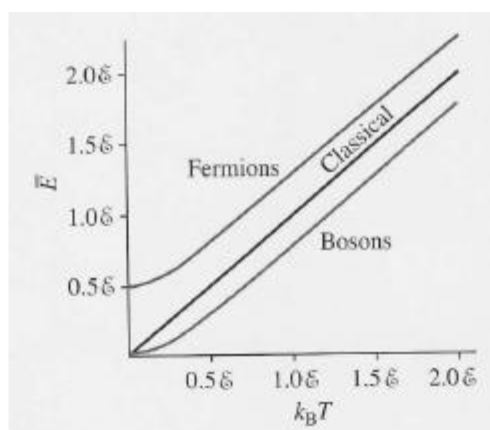
Distribution	Occupation index	Particles	Identical particles?	Spin	Distingui-shable?	Exclusion principle?	Examples
Boltzman	$\frac{1}{Be^{E/k_BT}}$	Classical	Yes	Any spin	Yes	No	Gas molecules
Bose-Einstein	$\frac{1}{Be^{E/k_BT}-1}$	Bosons	Yes	0 or integer spin	No	No	Photons in blackbody radiation
Fermi-Dirac	$\frac{1}{Be^{E/k_BT}+1}$	Fermions	Yes	1/2	No	Yes	Electrons in semiconductors

- In all three probability distributions, the probability of a particle occupying higher E state decreases.
- If $k_B T$ is larger than an energy state, particles tend to diffuse and probability of particles occupying the same energy state decreases. See the graph when $k_B T$= *5E*. There are almost no distinctions among the three types of distributions.
- If $k_B T$is smaller than an energy state, particles tend to pile up according to the rules allowed for the particular type of particles. See when $k_B T = \frac{1}{5} E$. Fermions the occupation number never exceeds 1. At low temperatures, bosons tend to congregate in the lowest-energy individual-particle state, while fermions tend to fill states, one particle per state, upto the some maximum energy.



(occupation number vs. E)

(Average energy vs. temperature)



- **At high temperatures (**$k_B T \gg E$**)**, both bosons and fermions follow the linear relationship of $\bar{E} = k_B T$, the same as the classical result. Still, energy is somewhat lower for bosons and higher for Fermions.
- **At low temperatures (**$k_B T \ll E$**)**, fermions' energy falls to ½ E while bosons and classical particles fall to zero.

Fermi Energy

- At low temperatures, the Fermi-Diract occupation number is nearly 1 to a certain energy, then drops zero suddenly. This energy is known as Fermi energy below which all the energy levels are filled up with fermions without vacancy at T=0. Mathematically, Fermi energy is defined as the energy of which occupation number becomes ½

$\mathcal{N}(E_F) = \frac{1}{Be^{E/k_B T}+1} = \frac{1}{2}$ which makes $B = e^{-\frac{E_F}{k_B T}}$

$$\mathcal{N}(E_F) = \frac{1}{e^{(E-E_F)/k_B T} + 1}$$

- $k_B T (when\ T = 300K, room\ temperature)$

$$= \left(1.38\text{x}10^{-23}\frac{J}{K}\right)(300K) = 4.14 \text{ x } 10^{-21} J = 0.026\ eV$$

- Occupation number vs. Energy: E_F can change according to temperature. However, E_F can be considered nearly constant in most daily applications.