PH102: Interactive Lecture 5

- Topics
  - 3d Schrodinger Equation for Hydrogen atom
  - $(x, y, z) \leftrightarrow (r, \theta, \phi)$
  - Separation of variables $R\Theta\Phi$
  - Three equations
  - Three quantum numbers
  - Wave functions
  - Quantization of angular momentum (L)

Hydrogen atom

- Potential created by Coulomb interactions between electron ($-e$) and proton ($+e$)

Hydrogen atom

- Potential created by Coulomb interactions between electron (-*e*) and proton (+*e*)

$$U(x) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

Hydrogen atom

- Potential created by Coulomb interactions between electron (-*e*) and proton (+*e*)

$$U(x) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

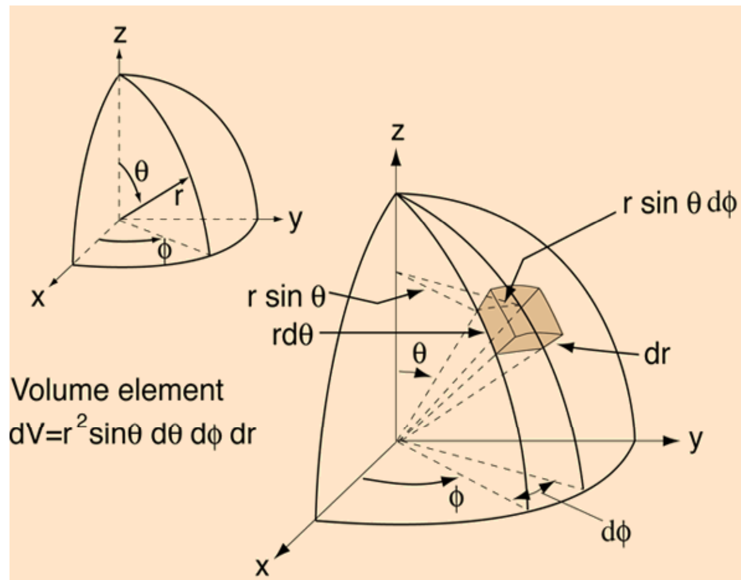
  - Symmetric in r
  - Choose (r, θ, ϕ)

$$(x, y, z) \quad \leftrightarrow \quad (r, \theta, \phi)$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \phi = tan^{-1}\dfrac{y}{x} \\ \theta = cos^{-1}\dfrac{z}{r} \end{cases}$$

$$\begin{cases} x = rcos\phi\ sin\vartheta \\ y = r\sin\phi\cos\theta \\ z = rcos\theta \end{cases}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$



$$(x, y, z) \quad \leftrightarrow \quad (r, \theta, \phi)$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \phi = tan^{-1}\frac{y}{x} \\ \theta = cos^{-1}\frac{z}{r} \end{cases}$$

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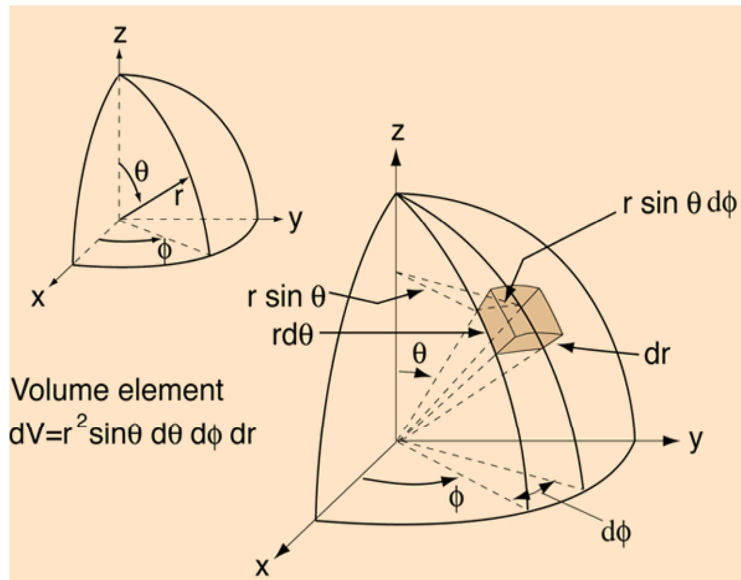

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\ \frac{\partial}{\partial\theta}\left(sin\theta\ \frac{\partial}{\partial\theta}\right) + csc^2\theta\ \frac{\partial}{\partial\phi^2}\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\ \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m} \nabla^2 \psi\ (\vec{x}) + U(\vec{x})\psi\ (\vec{x}) = E\ \psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + csc\theta\ \frac{\partial}{\partial \theta}\left(sin\theta\ \frac{\partial}{\partial \theta}\right) + csc^2\theta\ \frac{\partial}{\partial \phi^2}\right]$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + csc\theta\ \frac{\partial}{\partial \theta}\left(sin\theta\ \frac{\partial}{\partial \theta}\right) + csc^2\theta\ \frac{\partial^2}{\partial \phi^2}\right]\psi(r,\theta,\phi) + U(r)\ \psi(r,\theta,\phi) = E\ \psi(r,\theta,\phi)$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + csc^2\theta\,\frac{\partial}{\partial\phi^2}\right]$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + csc^2\theta\,\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\,\nabla^2\,\psi\,(\vec{x}) + U(\vec{x})\psi\,(\vec{x}) = E\;\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\;\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + \;csc^2\theta\;\frac{\partial}{\partial\phi^2}\right]$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\;\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + \;csc^2\theta\,\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\;\psi(r,\theta,\phi) = E\;\psi(r,\theta,\phi)$$

$$csc\theta\;\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right)\psi(r,\theta,\phi) + \;csc^2\theta\,\frac{\partial^2}{\partial\phi^2}\psi(r,\theta,\phi)$$

$$= \;[-\;\frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right) - \;\frac{2mr^2}{\hbar^2}\;(E - U(r))]\;\psi(r,\theta,\phi)$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + csc^2\theta\,\frac{\partial}{\partial\phi^2}\right]$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + csc^2\theta\,\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

$$csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right)\psi(r,\theta,\phi) + csc^2\theta\,\frac{\partial^2}{\partial\phi^2}\psi(r,\theta,\phi)$$

$$= \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(E - U(r)\right)\right]\psi(r,\theta,\phi)$$

Separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + csc^2\theta\,\frac{\partial}{\partial\phi^2}\right]$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + csc^2\theta\,\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

$$csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right)\psi(r,\theta,\phi) + csc^2\theta\,\frac{\partial^2}{\partial\phi^2}\psi(r,\theta,\phi)$$

$$= \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E - U(r))\right]\psi(r,\theta,\phi)$$

Separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

$$R\Phi\,csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right) + R\Theta csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2} = -\Theta\Phi\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E - U(r))\,R\Theta\Phi$$

Schrodinger Equation

$$\frac{1}{\Theta}\, csc\theta\, \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi} csc^2\theta\, \frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\,(E - U(r)) = C \text{ (Constant)}$$

Angular part

Radial part

Schrodinger Equation

$$\frac{1}{\Theta}\ csc\theta\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\,(E-U(r))=C\ (\text{Constant})$$

Angular part

Radial part

-l (l + 1)

Schrodinger Equation

$$\frac{1}{\Theta}\, csc\theta\, \frac{\partial}{\partial\theta}\left(sin\theta\, \frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi} csc^2\theta\, \frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(E - U(r)\right) = C \text{ (Constant)}$$

Angular part → ***-l (l + 1)***

Radial part → ***-l (l + 1)***

C (Constant) → ***-l (l + 1)***

Schrodinger Equation

$$\frac{1}{\Theta}\ csc\theta\ \frac{\partial}{\partial\theta}\left(sin\theta\ \frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\ (E - U(r)) = C\ (\text{Constant})$$

Angular part

Radial part

-l (l + 1)

-l (l + 1)

-l (l + 1)

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\ (E - U(r)) = C = -l(l+1) \\ \frac{1}{\Theta}\ csc\theta\ \frac{\partial}{\partial\theta}\left(sin\theta\ \frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2} = C = -l(l+1) \end{cases}$$

Schrodinger Equation

$$\frac{1}{\Theta}\ csc\theta\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C\ (\text{Constant})$$

Angular part

-l (l + 1)

Radial part

-l (l + 1)

-l (l + 1)

$$\begin{cases}-\dfrac{1}{R}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial R}{\partial r}\right)-\dfrac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C=-l(l+1)\\ \dfrac{1}{\Theta}\ csc\theta\ \dfrac{\partial}{\partial\theta}\left(sin\theta\,\dfrac{\partial\Theta}{\partial\theta}\right)+\dfrac{1}{\Phi}csc^2\theta\,\dfrac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1)\end{cases}$$

$$\frac{1}{\Theta}\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right)+l(l+1)sin^2\theta=-\frac{1}{\Phi}\ \frac{\partial^2\Phi}{\partial\phi^2}$$

$$=\ m_l^2\ (\text{another constant})$$

Schrodinger Equation

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi \\ sin\theta \dfrac{\partial}{\partial \theta}\left(sin\theta \dfrac{\partial \Theta}{\partial \theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0 \\ \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial R}{\partial r}\right) + \dfrac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0 \end{cases}$$

Schrodinger Equation

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi & \text{Azimuthal Equation} \\ \sin\theta \dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial \Theta}{\partial \theta}\right) + [l(l+1)\sin^2\theta - m_l^2]\Theta = 0 & \text{Polar Equation} \\ \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial R}{\partial r}\right) + \dfrac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0 & \text{Radial Equation} \end{cases}$$

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function

Boundary condition

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\Phi(\phi) = A\, e^{i m_l \phi}$

Boundary condition

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Azimuthal Equation

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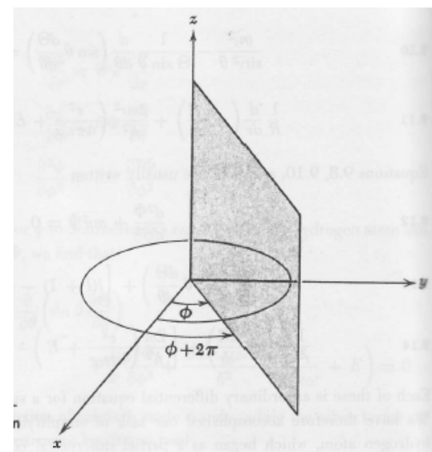
Boundary condition



Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $$\Phi(\phi) = A\, e^{i m_l \phi}$$

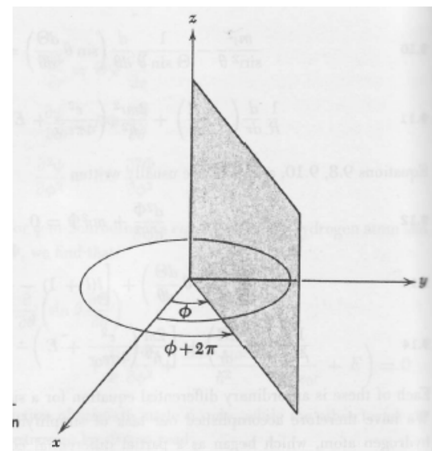
Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

$$A\, e^{i m_l \phi} = A\, e^{i m_l (\phi + 2\pi)} = A e^{i m_l \phi} e^{i 2\pi m_l}$$

$$e^{i 2\pi m_l} = 1 = cos 2\pi m_l + \; i sin 2\pi m_l$$

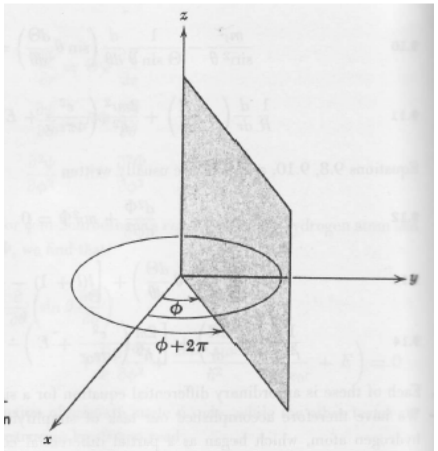
Quantization



Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\Phi(\phi) = A\, e^{im_l\phi}$



Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

$$A\, e^{im_l\phi} = A\, e^{im_l(\phi+2\pi)} = Ae^{im_l\phi}e^{i2\pi m_l}$$

$$e^{i2\pi m_l} = 1 = cos2\pi m_l + isin2\pi m_l$$

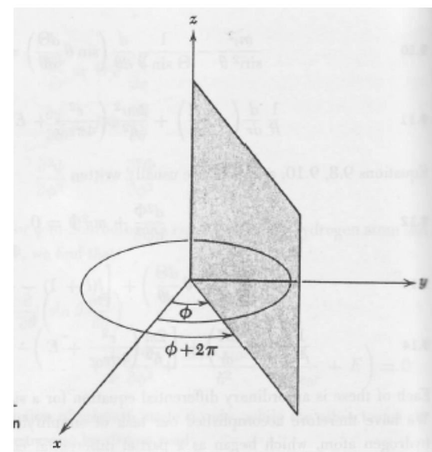
Quantization

$$m_l = 0, \pm1, \pm2, \pm3, etc.$$

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\quad \Phi(\phi) = A\, e^{i m_l \phi}$



Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

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$$e^{i 2\pi m_l} = 1 = cos 2\pi m_l + \; i sin 2\pi m_l$$

Quantization

$$m_l = 0, \pm 1, \pm 2, \pm 3, etc.$$

Magnetic quantum number

Polar Equation

$$sin\theta \frac{\partial}{\partial\theta}\left(sin\theta \frac{\partial\Theta}{\partial\theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0$$

Solutions: Associated Legendre Functions
Quantization

any given l, m_l $values\ can\ be\ 0, \pm 1, \pm 2, \pm l$

→ **Orbital quantum number**

Radial Equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0$$

Solutions: associated Laguerre functions
Quantization:

$$E_n = -\frac{me^4}{32\pi\,\varepsilon_0^2\hbar^2}\left(\frac{1}{n^2}\right) \text{ where } n \text{ is an integer}$$

n → **Principal quantum number**

l = 0, 1, 2, …(*n*-1)

Solutions

- Principal quantum number, $n = 1, 2, 3, \ldots$
- Orbital quantum number, $l = 0, 1, 2, \ldots (n-1)$ where l =1*(s)*, =2*(p)*, =3*(d)*, =4*(f)*, *etc.*
- Magnetic quantum number, $m_l = 0, \pm 1, \pm 2, \ldots \pm l$

$$\text{Wave function} = \psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l}$$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

Symbolic designation of atomic states

	s $l = 0$	*p* $l = 1$	*d* $l = 2$	*f* $l = 3$	*g* $l = 4$	*h* $l = 5$
$n = 1$	*1s*					
$n = 2$	*2s*	*2p*				
$n = 3$	*3s*	*3p*	*3d*			
$n = 4$	*4s*	*4p*	*4d*	*4f*		
$n = 5$	*5s*	*5p*	*5d*	*5f*	*5g*	
$n = 6$	*6s*	*6p*	*6d*	*6f*	*6g*	*6h*

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,(E-U(r))R\,-l(l+1)R=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,[E-U(r)\,-\frac{l(l+1)\hbar^2}{2mr^2}]R=0$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}\,(E - U(r))R \;- l(l+1)R = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}\,[E - U(r) \;- \frac{l(l+1)\hbar^2}{2mr^2}]R = 0$$

E = Kinetic E (radial) + Kinetic E (orbital) + $U(r)$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)R-l(l+1)R=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left[E-U(r)-\frac{l(l+1)\hbar^2}{2mr^2}\right]R=0$$

$$\mathrm{E} = \text{Kinetic E (radial)} + \text{Kinetic E (orbital)} + U(r)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left[Kineric\ E\ (radial)+\ Kinetic\ E\ (orbital)-\frac{l(l+1)\hbar^2}{2mr^2}\right]R=0$$

$$Kinetic\ E\ (orbital)=\frac{l(l+1)\hbar^2}{2mr^2}$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)R-l(l+1)R=0$$

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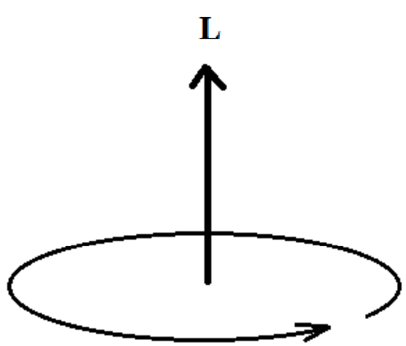
$$\text{E} = \text{Kinetic E (radial)} + \text{Kinetic E (orbital)} + U(r)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left[Kineric\ E\ (radial)+\ Kinetic\ E\ (orbital)-\frac{l(l+1)\hbar^2}{2mr^2}\right]R=0$$

$$Kinetic\ E\ (orbital)=\frac{l(l+1)\hbar^2}{2mr^2}$$



$$\frac{1}{2}m\,v_{orbital}^2=\frac{L^2}{2mr^2}$$

$$L=mv_{orbital}r \rightarrow v_{orbital}=\frac{L}{mr}$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,(E-U(r))R\;-l(l+1)R=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,[E-U(r)\;-\frac{l(l+1)\hbar^2}{2mr^2}]R=0$$

$$\mathrm{E} = \text{Kinetic E (radial)} + \text{Kinetic E (orbital)} + U(r)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\,[Kineric\ E\ (radial)+\ Kinetic\ E\ (orbital)\;-\frac{l(l+1)\hbar^2}{2mr^2}]R=0$$

$$Kinetic\ E\ (orbital)=\frac{l(l+1)\hbar^2}{2mr^2}$$

$$\frac{1}{2}m\,v_{orbital}^2=\frac{L^2}{2mr^2}$$

$$L=mv_{orbital}r \rightarrow v_{orbital}=\frac{L}{mr}$$

$$\frac{L^2}{2mr^2}=\frac{l(l+1)\hbar^2}{2mr^2}$$

$$L^2=\ l(l+1)\hbar^2$$

Origin of angular momentum quantization

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)R-l(l+1)R=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left[E-U(r)-\frac{l(l+1)\hbar^2}{2mr^2}\right]R=0$$

$$\text{E} = \text{Kinetic E (radial)} + \text{Kinetic E (orbital)} + U(r)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left[Kineric\ E\ (radial)+\ Kinetic\ E\ (orbital)-\frac{l(l+1)\hbar^2}{2mr^2}\right]R=0$$

$$Kinetic\ E\ (orbital)=\frac{l(l+1)\hbar^2}{2mr^2}$$

$$\frac{1}{2}m\,v_{orbital}^2=\frac{L^2}{2mr^2}$$

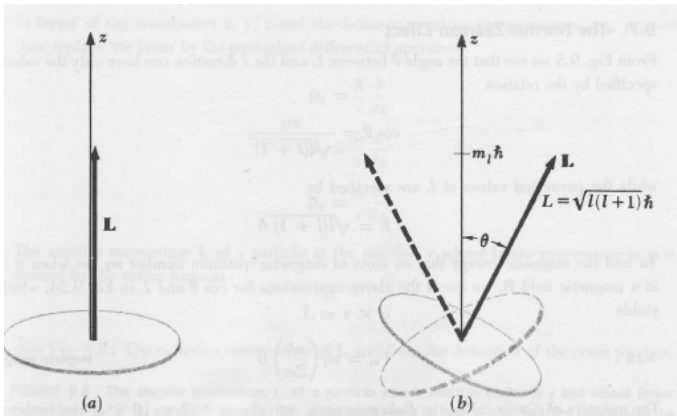
$$L=mv_{orbital}r \rightarrow v_{orbital}=\frac{L}{mr}$$

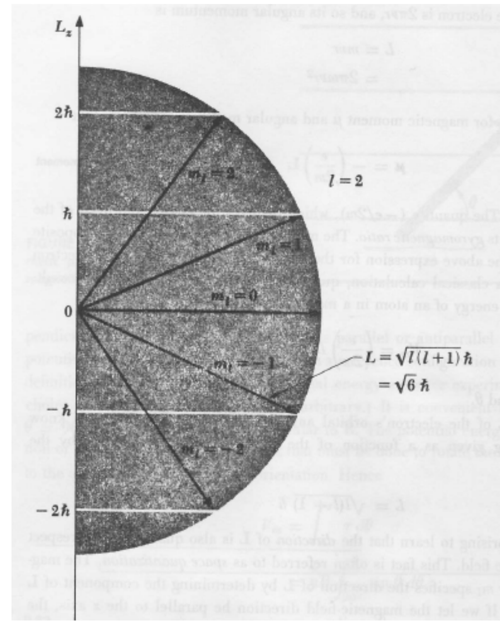
$$\frac{L^2}{2mr^2}=\frac{l(l+1)\hbar^2}{2mr^2}$$

$$L^2=\ l(l+1)\hbar^2$$

Angular momentum is quantized

Angular Momentum (L)

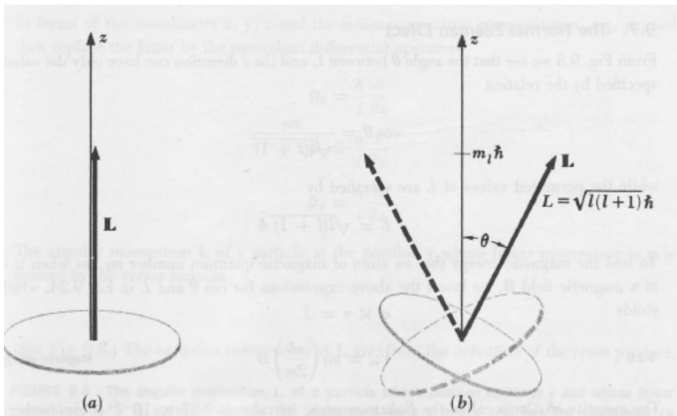


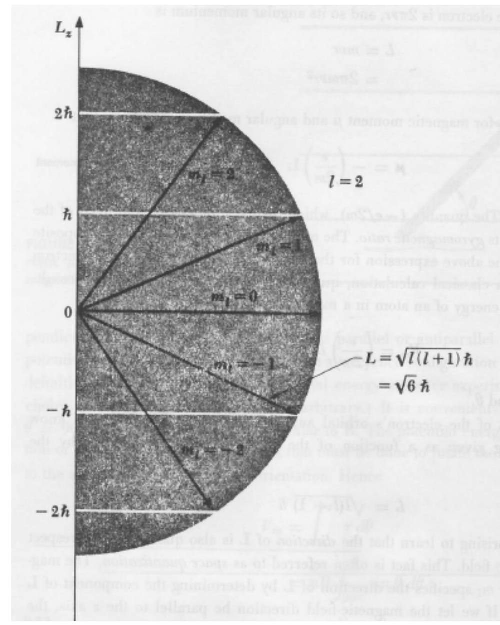


Amount =

Direction =

Angular Momentum (L)





Amount = $L = \sqrt{l(l+1)}\hbar$

Direction = the m_l *value* determines the direction of L